Economics 511 Problem Set 15 Answers

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1. Suppose $f(x)$ is a concave function. Show that $g(x) = -f(x)$ is a convex function.

   By definition (page 53 in HLMRS), a function $f : X \longmapsto Y$, where X is the convex domain, is concave if and only if

$$f(\lambda x' + (1-\lambda)x'') \geq \lambda f(x') + (1-\lambda)f(x'')$$

   for all x' and x'' in X and for all $\lambda, 0 \leq \lambda \leq 1$. Multiplying this inequality by -1 yields

$$-f(\lambda x' + (1-\lambda)x'') \leq -\lambda f(x') + (1-\lambda)(-1) f(x'')$$

   or

$$g(\lambda x' + (1-\lambda)x'') \leq \lambda g(x') + (1-\lambda)g(x'')$$

   and so $g(x) = -f(x)$ is a convex function.

2. Suppose $g(x)$ is a convex function. Show that $f(x) = -g(x)$ is a concave function.

   Since

$$g(\lambda x' + (1-\lambda)x'') \leq \lambda g(x') + (1-\lambda)g(x'')$$

   then

$$-f(\lambda x' + (1-\lambda)x'') \leq -\lambda f(x') + (1-\lambda)(-1) f(x'')$$

   or

$$f(\lambda x' + (1-\lambda)x'') \geq \lambda f(x') + (1-\lambda)f(x'')$$

   and hence $f(x) = -g(x)$ is a concave function.

3. Suppose $f(x)$ is a function on a convex domain $X \subseteq \mathbb{R}^n$. Show that

   (a) f is concave if and only if

$$G^+ = \{(x,y) : x \in X \text{ and } f(x) \geq y\} \text{ is a convex set}$$

   Proof: Suppose f is concave and that (x', y') and (x'', y'') are in G^+. Then $f(x') \geq y'$ and $f(x'') \geq y''$. Moreover, since f is concave

$$\begin{aligned} f(\lambda x' + (1-\lambda)x'') &\geq \lambda f(x') + (1-\lambda)f(x'') \\ &\geq \lambda y' + (1-\lambda)y'' \end{aligned}$$

for all $x', x'' \in X$ and all $\lambda \in (0,1)$. We conclude that $\lambda(x',y') + (1-\lambda)(x'',y'')$ is in G^+ and hence G^+ is a convex set.
Conversely, suppose $f(x') = y'$ and $f(x'') = y''$. Then $(x',y') \in G^+$ and $(x'',y'') \in G^+$. Since G^+ is a convex set,

$$\lambda(x',y') + (1-\lambda)(x'',y'') \in G^+.$$

This implies that

$$\begin{aligned} f(\lambda x' + (1-\lambda)x'') &\geq \lambda y' + (1-\lambda)y'' \\ &= \lambda f(x') + (1-\lambda)f(x'') \end{aligned}$$

and thus f is a concave function.

(b) f is a convex function if and only if

$$G^- = \{(x,y) : x \in X \text{ and } f(x) \leq y\} \text{ is a convex set.}$$

The answer to (b) is analogous to the answer to (a) and is left as an exercise.

4. Let $h(x)$ be a linear combination of a collection of functions, $g^k(x), k = 1, ..., K$, i.e.,

$$h(x) = \sum_{k=1}^{K} a_k g^k(x).$$

Assuming that $a_k \geq 0, k = 1, ..., K$, show that

(a) $h(x)$ is a concave function if each $g^k, k = 1, ..., K$, is a concave function.
If each g^k is concave then

$$g^k\left(\lambda x' + (1-\lambda)x''\right) \geq \lambda g^k(x') + (1-\lambda)g^k(x''), k = 1, ..., K.$$

Multiply by $a_k \geq 0$ to get

$$a_k g^k\left(\lambda x' + (1-\lambda)x''\right) \geq \lambda a_k g^k(x') + (1-\lambda)a_k g^k(x''), k = 1, ..., K.$$

Sum over k to get

$$\sum_{k=1}^{K} a_k g^k\left(\lambda x' + (1-\lambda)x''\right) \geq \lambda \sum_{k=1}^{K} a_k g^k(x') + (1-\lambda) \sum_{k=1}^{K} a_k g^k(x''), k = 1, ..., K,$$

i.e.

$$h\left(\lambda x' + (1-\lambda)x''\right) \geq \lambda h(x') + (1-\lambda)h(x'')$$

and hence h is concave.

(b) $h(x)$ is a convex function if each $g^k, k = 1, ..., K$, is a convex function.

Just repeat the argument in (a) with $\geq$ replaced by $\leq$.

5. Your textbook gives the following definition of a quasiconcave function. A function f with convex domain $X \subseteq \mathbb{R}^n$ is quasiconcave if, for every point $x^0 \in X$, the better set

$$B(x^0) = \{x \in X : f(x) \geq f(x^0)\}$$

is a convex set. Show that the above definition is equivalent to the following definition. A function f with convex domain $X \subseteq \mathbb{R}^n$ is quasiconcave if for every x' and x'' in X, if $f(x') \geq f(x'')$ then $f(\lambda x' + (1-\lambda)x'') \geq f(x'')$.

Proof: Suppose $f(x') \geq f(x'')$. Then, by definition, $x', x'' \in B(x'')$. If $B(x'')$ is a convex set then $\lambda x' + (1-\lambda)x'' \in B(x'')$ and hence $f(\lambda x' + (1-\lambda)x'') \geq f(x'')$. On the other hand if $f(\lambda x' + (1-\lambda)x'') \geq f(x'')$ then $\lambda x' + (1-\lambda)x'' \in B(x'')$ and thus $B(x'')$ is a convex set.

6. Let $A \subseteq \mathbb{R}^n$be a convex set and consider

$$\max f(x) \text{ subject to } x \in A.$$

Suppose f is a quasiconcave function. Show that the set of global maximizers

$$Z = \{z : z \in A, f(z) \geq f(x) \text{ for all } x \in A\}$$

is a convex set.

Proof: Suppose $z', z'' \in Z$. Clearly, $f(z') = f(z'')$. Moreover, by quasiconcavity,

$$f(\lambda z' + (1-\lambda)z'') \geq f(z') = f(z'').$$

We conclude that $\lambda z' + (1-\lambda)z'' = \bar{z} \in Z$ and hence Z is a convex set.

7. Your textbook states that a function f is strictly quasiconcave if each better set is strictly convex. Here is an equivalent definition that I prefer. A function f with convex domain $X \subseteq \mathbb{R}^n$ is strictly quasiconcave if for every x' and x'' in X, if $f(x') \geq f(x'')$ then $f(\lambda x' + (1-\lambda)x'') > f(x'')$ for all $\lambda, 0 < \lambda < 1$. Prove that:

(a) If f is a strictly quasiconcave function then any local maximizer of f on a convex feasible set A is also a global maximizer of f on A.

Proof: Suppose $x^* \in A$ is a local maximizer of f on A. Then there exists $\epsilon > 0$ such that

$$f(x^*) \geq f(x) \text{ for all } x \in N_\epsilon(x^*) \cap A.$$

If $z \in Z \subset A$ is a global maximizer and x^* is not a global maximizer then $f(z) > f(x^*)$. It follows that $x(\lambda) = \lambda x^* + (1-\lambda)z \in A$ [since A is convex] and, by strict quasiconcavity,

$$f(\lambda x^* + (1-\lambda)z) = f(x(\lambda)) > f(x^*),$$

for every $\lambda \in (0,1)$. For

$$1 - \frac{\epsilon}{\|x^* - z\|} < \lambda < 1,$$

we have $\|x^* - x(\lambda)\| < \epsilon$ so that $x(\lambda) \in N_\epsilon(x^*) \cap A$. This contradicts the assumption that x^* is a local maximizer.

(b) If f is a strictly quasiconcave function then the set of global mazimizers Z consists of a single point, i.e., the global maximizer is unique.
Proof: Suppose $z, z' \in Z$ and $z \neq z'$. Then $f(z') = f(z) \Rightarrow f(\lambda z + (1-\lambda)z') > f(z) = f(z')$ by strict quasiconcavity. Since $\lambda z + (1-\lambda)z' \in A$ this contradicts the presumption that $z, z' \in Z$. Hence, $z = z'$.

8. For each of the following problems find the solutions that satisfy the appropriate Kuhn-Tucker (KT) conditions. The solution includes values of the multipliers as well as values of the choice variables.

(a)

$$\begin{array}{rl} \text{maximize} & \ln x_1 + \ln x_2 \\ \text{subject to} & x_1 + x_2 \leq 25, \quad x_1 \leq 10 \\ & x_1, x_2 \geq 0 \end{array}$$

$$L = \ln x_1 + \ln x_2 + \lambda(25 - x_1 - x_2) + \kappa(10 - x_1)$$

$$\begin{aligned} L_1 &= \frac{1}{x_1} - \lambda - \kappa \leq 0, x_1 \geq 0, \left(\frac{1}{x_1} - \lambda - \kappa\right) x_1 = 0 \quad (1) \\ L_2 &= \frac{1}{x_2} - \lambda \leq 0, x_2 \geq 0, \left(\frac{1}{x_2} - \lambda\right) x_2 = 0 \quad (2) \\ L_\lambda &= 25 - x_1 - x_2 \geq 0, \lambda \geq 0, (25 - x_1 - x_2)\,\lambda = 0 \quad (3) \\ L_\kappa &= 10 - x_1 \geq 0, \kappa \geq 0, (10 - x_1)\,\kappa = 0 \quad (4) \end{aligned}$$

We must have $x_1 > 0$ and $x_2 > 0$.
Case 1: $\lambda > 0, \kappa > 0$. Then $(4): 10 - x_1 = 0$ and $(3): 25 - x_1 - x_2 = 0$. Hence $(x_1, x_2) = (10, 15)$. Then $(2): \lambda = 1/15$. and $(1): 1/10 - 1/15 - \kappa = 0$, and so $\kappa = 1/30$. Then $\ln x_1 + \ln x_2 = \ln(10) + \ln(15)$.
Case 2: $\lambda > 0, \kappa = 0$. Then $(3): 25 - x_1 - x_2 = 0$ and $(4)\ 10 - x_1 \geq 0$. However, from $(1), (2)$ and (3) , $x_1 = x_2 = 12.5$, a contradiction.
Case 3: $\lambda = 0, \kappa > 0$. By (2), $1/x_2 \leq 0$ which contradicts $x_2 > 0$.

(b)

$$\begin{array}{rl} \text{minimize} & x_1 + x_2 \\ \text{subject to} & x_1 x_2 \geq 100, \quad x_1 \geq 5 \\ & x_1, x_2 \geq 0 \end{array}$$

The Lagrangian for this problem is:

$$L = x_1 + x_2 + \lambda \left(100 - x_1 x_2\right) + \kappa \left(5 - x_1\right)$$

Unfortunately, the function $g^! \left(x_1, x_2\right) = 100 - x_1 x_2$ is not a convex function; it is only quasiconvex. This case can still be handled if one appeals to results by Kenneth Arrow and Alain Enthoven, "Quasi-Concave Programming," *Econometrica*, Vol. 29, No. 4, (Oct., 1961), pp. 779-800. However, if we want to apply the results in the textbook we can accomplish this by taking the square root of the constraint to get the equivalent condition, $x_1^{1/2} x_2^{1/2} \geq 10$.The modified function, $\hat{g}^1 \left(x_1, x_2\right) = 10 - x_1^{1/2} x_2^{1/2}$ is a convex function and the equivalent constrained minimization problem is:

$$\begin{array}{rl} \text{minimize} & x_1 + x_2 \\ \text{subject to} & x_1^{1/2} x_2^{1/2} \geq 10, \quad x_1 \geq 5 \\ & x_1, x_2 \geq 0 \end{array}$$

with Lagrangian:

$$L = x_1 + x_2 + \lambda \left(10 - x_1^{1/2} x_2^{1/2}\right) + \kappa \left(5 - x_1\right)$$

$$\begin{aligned} L_1 &= 1 - \lambda \left(\frac{1}{2} x_1^{-1/2} x_2^{1/2}\right) - \kappa \geq 0, \\ & \quad x_1 \geq 0, \left(1 - \lambda \left(\frac{1}{2} x_1^{-1/2} x_2^{1/2}\right) - \kappa\right) x_1 = 0 \quad (1) \\ L_2 &= 1 - \lambda \left(\frac{1}{2} x_1^{1/2} x_2^{-1/2}\right) \geq 0, \\ & \quad x_2 \geq 0, \left(1 - \lambda \left(\frac{1}{2} x_1^{1/2} x_2^{-1/2}\right)\right) x_2 = 0 \quad (2) \\ L_\lambda &= 10 - x_1^{1/2} x_2^{1/2} \leq 0, \lambda \geq 0, \left(100 - x_1^{1/2} x_2^{1/2}\right) \lambda = 0 \quad (3) \\ L_\kappa &= 5 - x_1 \leq 0, \kappa \geq 0, \left(5 - x_1\right) \kappa = 0 \quad (4) \end{aligned}$$

From (3), $x_1^{1/2} x_2^{1/2} \geq 10$. Since $x_1, x_2 \geq 0$ this requires that $x_1 > 0$ and $x_2 > 0$.

Case 1: $\lambda > 0$ and $\kappa > 0$. By (3) : $x_1 x_2 = 100$ and by (4) : $x_1 = 5$. Thus $x_2 = 20$. Using (2) we get

$$\begin{aligned} 1 - \lambda\left(\frac{1}{2}\left(\frac{x_1}{x_2}\right)^{1/2}\right) &= 1 - \lambda\left(\frac{1}{2}\left(\frac{1}{4}\right)^{1/2}\right) \\ &= 1 - \frac{\lambda}{4} = 0 \\ \lambda &= 4. \end{aligned}$$

Using (1) we get

$$\begin{aligned} 1 - 4\left(\frac{1}{2}\left(\frac{x_2}{x_1}\right)^{1/2}\right) - \kappa &= 1 - 4\left(\frac{1}{2}(4)^{1/2}\right) - \kappa \\ &= 1 - 4 - \kappa = 0 \end{aligned}$$

Hence, $\kappa = -3$, a contradiction.
Case 2: $\lambda > 0, \kappa = 0$. From $(1), (2)$ and (3) we get $x_1 = x_2 = 10$. Hence, $\lambda = 2$. Finally, $x_1 + x_2 = 10 + 10 = 20$.
Case 3: $\lambda = 0, \kappa = 0$. However, from (2) we get $(1 - 0)\, x_2 = x_2 = 0$, a contradiction.

9. Consider the problem:

$$\max x_1 + \ln x_2 \text{ subject to } m - px_1 - x_2 \geq 0, x_1 \geq 0, x_2 \geq 0,$$

where $p > 0$ and $m > 0$. Verify that the objective function is a concave function. Find and *characterize* all possible solutions to this problem.

First we verify that the objective function is concave, i.e., that $f(x_1, x_2) = x_1 + \ln x_2$ s a concave function. We compute the Hessian matrix.

$$\begin{aligned} f_1 &= 1, f_2 = x_2^{-1} \\ f_{11} &= 0, f_{12} = f_{21} = 0, f_{22} = -x_2^{-2} = \frac{-1}{x_2^2}. \end{aligned}$$

$$H = \begin{bmatrix} 0 & 0 \\ 0 & \frac{-1}{x_2^2} \end{bmatrix}$$

All minors are nonpositive: f is concave.

$$\begin{aligned} \mathcal{L} &= x_1 + \ln x_2 + \lambda(m - px_1 - x_2) \\ \mathcal{L}_1 &= 1 - \lambda p \leq 0, x_1 \geq 0, (1 - \lambda p)\, x_1 = 0 \qquad 1(a)(b)(c) \\ \mathcal{L}_2 &= \frac{1}{x_2} - \lambda \leq 0, x_2 \geq 0, \left(\frac{1}{x_2} - \lambda\right) x_2 = 0 \qquad 2(a)(b)(c) \\ \mathcal{L}_\lambda &= m - px_1 - x_2 \geq 0, \lambda \geq 0, (m - px_1 - x_2)\, \lambda = 0 \qquad 3(a)(b)(c) \end{aligned}$$

Is $\lambda = 0$? No, because this gives $1 \leq 0$ from $(1)(a)$. Henceforth, $\lambda > 0$ and $m - px_1 - x_2 = 0$.
Case 1: $x_1 > 0, x_2 > 0$. Then $1 - \lambda p = 0$ or $\lambda = 1/p$. Then, from $(2)(a)$, $x_2 = p$. Then, since $m - px_1 - x_2 = 0$ we get $m - px_1 - p = 0$. Solve to get $x_1 = \dfrac{m-p}{p}$. This requires that $m > p$.
Case 2: $x_1 = 0$, $x_2 > 0$. Then $x_2 = m, \lambda = 1/m$. Since $1 - p/m \leq 0$, we get $1 \leq p/m$, $m \leq p$.
Case 3: $x_2 = 0$. This is not admissible since when $x_2 \to 0, 1/x_2 \to +\infty$ violating $(2)(a)$.