Economics 511 Problem Set 15

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1. Suppose $f(x)$ is a concave function. Show that $g(x) = -f(x)$ is a convex function.

2. Suppose $g(x)$ is a convex function. Show that $f(x) = -g(x)$ is a concave function.

3. Suppose $f(x)$ is a function on a convex domain $X \subseteq \mathbb{R}^n$. Show that

   (a) f is a concave function if and only if

   $$G^+ = \{(x, y) : x \in X \text{ and } f(x) \geq y\} \text{ is a convex set.}$$

   (b) f is a convex function if and only if

   $$G^- = \{(x, y) : x \in X \text{ and } f(x) \leq y\} \text{ is a convex set.}$$

4. Let $h(x)$ be a linear combination of a collection of functions, $g^k(x), k = 1, ..., K$, i.e.,

   $$h(x) = \sum_{k=1}^{K} a_k g^k(x).$$

   Assuming that $a_k \geq 0, k = 1, ..., K$, show that

   (a) $h(x)$ is a concave function if each $g^k, k = 1, ..., K$, is a concave function.

   (b) $h(x)$ is a convex function if each $g^k, k = 1, ..., K$, is a convex function.

5. Your textbook gives the following definition of a quasiconcave function. A function f with convex domain $X \subseteq \mathbb{R}^n$ is quasiconcave if, for every point $x^0 \in X$, the better set

   $$B(x^0) = \left\{x \in X : f(x) \geq f(x^0)\right\}$$

   is a convex set. Show that the above definition is equivalent to the following definition. A function f with convex domain $X \subseteq \mathbb{R}^n$ is quasiconcave if for every x' and x'' in X, if $f(x') \geq f(x'')$ then $f(\lambda x' + (1 - \lambda)x'') \geq f(x'')$.

6. Let $A \subseteq \mathbb{R}^n$be a convex set and consider

   $$\max f(x) \text{ subject to } x \in A.$$

   Suppose f is a quasiconcave function. Show that the set of global maximizers

   $$Z = \{z : z \in A, f(z) \geq f(x) \text{ for all } x \in A\}$$

   is a convex set.

7. Prove that:

   (a) If f is a strictly quasiconcave function then any local maximizer of f on a convex feasible set A is also a global maximizer of f on A.

   (b) If f is a strictly quasiconcave function then the set of global maximizers Z consists of a single point, i.e., the global maximizer is unique.

8. For each of the following problems find the solutions that satisfy the appropriate Kuhn-Tucker (KT) conditions. The solution includes values of the multipliers as well as values of the choice variables.

   (a)
$$\begin{array}{rl} \text{maximize} & \ln x_1 + \ln x_2 \\ \text{subject to} & x_1 + x_2 \leq 25, \quad x_1 \leq 10 \\ & x_1, x_2 \geq 0 \end{array}$$

   (b)
$$\begin{array}{rl} \text{minimize} & x_1 + x_2 \\ \text{subject to} & x_1 x_2 \geq 100, \quad x_1 \geq 5 \\ & x_1, x_2 \geq 0 \end{array}$$

9. Consider the problem:

$$\max x_1 + \ln x_2 \text{ subject to } m - px_1 - x_2 \geq 0, x_1 \geq 0, x_2 \geq 0,$$

   where $p > 0$ and $m > 0$. Verify that the objective function is a concave function. Find and *characterize* all possible solutions to this problem.