Economics 511 Problem Set 16 Answers

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1. Evaluate the following integrals.

a) $\int 3x^2 + 2x + 1\,dx$ b) $\int e^{2x}\,dx$ c) $\int \frac{6x}{x^2+1}\,dx$

d) $\int x^{1/2}\,dx, \quad F(0) = 5$ e) $\int MC(Q)\,dQ, \quad MC(Q) = 10 + 4Q.$

(a)

$$\int 3x^2 + 2x + 1\,dx = x^3 + x^2 + x + C$$

(b)

$$\int e^{2x}\,dx = \frac{1}{2}e^{2x} + C$$

(c)

$$\begin{aligned} \int \frac{6x}{x^2+1}\,dx &= 3\int \frac{2x}{x^2+1}\,dx \\ &= 3\ln\left(x^2+1\right) + C \end{aligned}$$

(d)

$$\begin{aligned} F(x) &= \int x^{1/2}\,dx \\ &= \frac{2}{3}x^{3/2} + C \end{aligned}$$

$$\begin{aligned} F(0) &= C = 5 \\ F(x) &= \frac{2}{3}x^{3/2} + 5 \end{aligned}$$

(e)

$$\begin{aligned} TC(Q) &= \int 10 + 4Q\,dQ = 10Q + 2Q^2 + C \\ TC(0) &= C \\ TC(Q) &= 10Q + 2Q^2 + TC(0) \end{aligned}$$

2. The marginal product of labor is given by

$$MP(L) = abL^{b-1}, b > 0,$$

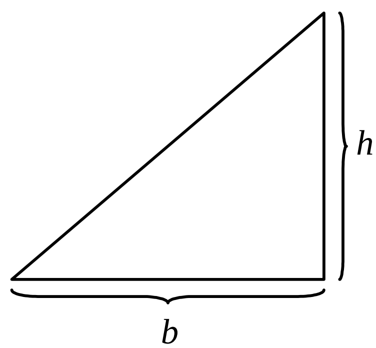
and the production function is $Q = F(L)$. Assume that $F(0) = 0$. Find the production function. For what values of b does the Law of Diminishing Marginal Returns hold?

$$F(L) = \int abL^{b-1}\, dL = aL^b + C$$

$F(0) = C = 0 \Rightarrow F(L) = aL^b$. The Law of Diminishing Marginal Returns implies that

$$\frac{dMP(L)}{dL} = ab(b-1)L^{b-2} < 0 \Rightarrow 0 < b < 1.$$

3. Find the area of the following triangle using integral calculus.



Hint: Evaluate the integral

$$\int_0^b \frac{h}{b} x\, dx$$

$$\int_0^b \frac{h}{b} x\, dx = \left.\frac{hx^2}{2b}\right]_0^b = \frac{1}{2}bh.$$

4. Evaluate the integrals

$$\text{a)} \int_0^1 3x^2\, dx \qquad \text{b)} \int_0^9 x^{1/2}\, dx$$

(a)

$$\int_0^1 3x^2\, dx = \left. x^3\right]_0^1 = 1$$

(b)

$$\int_0^9 x^{1/2}\, dx = \left.\frac{2}{3}x^{3/2}\right]_0^9 = \frac{2}{3}(9)^{3/2} = 18.$$

5. Evaluate the integrals

$$\text{a)}\ \int_0^{\infty} Ae^{-rt}\,dt \qquad \text{b)}\ \int_0^1 \frac{dx}{x}$$

(a)

$$\int_0^b Ae^{-rt}\,dt = \left.\frac{-Ae^{-rt}}{r}\right]_0^b = \frac{-Ae^{-rb}}{r} - \left(\frac{-Ae^{-0}}{r}\right)$$

$$\lim_{b\to+\infty}\left(\frac{Ae^{-0}}{r} - \frac{Ae^{-rb}}{r}\right) = \frac{A}{r}.$$

(b)

$$\int\limits_a^1 \frac{dx}{x} = \ln x]_a^1 = \ln 1 - \ln a = -\ln a$$

$$\lim_{a\to 0}(-\ln a) = +\infty$$

6. Consider

$$\int_a^c x^b\,dx, \quad 0 \le a < c.$$

(a) Find the range of values of b for which this integral converges as $a \to 0^+$.
First note that

$$\int_a^c x^{-1}\,dx = \ln c - \ln a.$$

This diverges as $a \to 0^+$. It also diverges as $c \to +\infty$. Hereafter assume $b \neq -1$. Now

$$\int_a^c x^b\,dx = \left[\frac{x^{b+1}}{b+1}\right]_a^c = \frac{c^{b+1}}{b+1} - \frac{a^{b+1}}{b+1}.$$

If $b+1 < 0$ then $a^{b+1} \to \infty$ as $a \to 0^+$. Hence, this integral converges only for $b+1 > 0$.

(b) Find the range of values of b for which this integral converges as $c \to +\infty$ (when $a > 0$)
Convergence requires that $b + 1 < 0$.

(c) Prove that $\int_0^{+\infty} x^b\,dx$ diverges for all values of b.
Done.

7. Evaluate the following integrals.

$$\text{a)}\ \int \frac{6x}{x^2+1}\,dx \qquad \text{b)}\ \int \frac{x}{\sqrt{1+x}}\,dx$$

(a) This was answered in problem 1 part c) using the rule that

$$\int \frac{f'(x)}{f(x)} dx = \ln f(x) + C.$$

Here we solve it by using the substitution rule. Let $u = x^2 + 1$. Then $du = 2x\, dx$. Hence

$$\begin{aligned} \int \frac{6x}{x^2+1}\, dx &= 3\int \frac{2x}{x^2+1}\, dx \\ &= 3\int \frac{1}{u}\, du \\ &= 3\ln u + C \\ &= 3\ln\left(x^2+1\right) + C \end{aligned}$$

(b) Let $u = x$ and $dv = (1+x)^{-1/2}dx$. Then $du = dx$ and $v = 2(1+x)^{1/2}$. Thus

$$\begin{aligned} \int \frac{x}{\sqrt{1+x}}\, dx &= \int x\,(1+x)^{-1/2}\, dx \\ &= \int u\, dv \\ &= uv - \int v\, du \\ &= x\left[2(1+x)^{1/2}\right] - \int 2(1+x)^{1/2}\, dx \\ &= 2x(1+x)^{1/2} - \frac{4}{3}(1+x)^{3/2} + C \end{aligned}$$