Economics 511 Problem Set 22 Answers
Nonlinear First-Order Differential Equations Answers

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1. (Exercise 1, page 887) Use a phase diagram and theorem 22.2 to conduct a qualitative analysis of $\dot{y} = -y + y^2 + 3/16$.

$$y^2 - y + \frac{3}{16} \Rightarrow \left(y - \frac{1}{4}\right)\left(y - \frac{3}{4}\right) = 0$$

$$\bar{y} = \left(\frac{1}{4}, \frac{3}{4}\right)$$

$$\frac{d\dot{y}}{dy} = 2y - 1 = \left(-\frac{1}{2}, \frac{1}{2}\right) = (\text{stable, unstable})$$

2. (Exercise 5, page 887) Quantity demanded in a market is given by

$$q^d = p^{-2}$$

and quantity supplied is given by

$$q^s = 8p.$$

If price adjusts according to $\dot{p} = \alpha(q^d - q^s)$, where $\alpha > 0$ is a constant, conduct a qualitative analysis of the dynamics of market price.

The differential equation is

$$\dot{p} = \alpha\left(p^{-2} - 8p\right).$$

The steady-state values are

$$p^{-2} = 8p; 1 = 8p^3; \bar{p} = \left(\frac{1}{8}\right)^{1/3}; \bar{p} = \frac{1}{2}.$$

$$\left.\frac{d\dot{p}}{dp}\right|_{p=1/2} = \alpha\left(-2p^{-3} - 8\right)\Big|_{p=1/2} = -24\alpha < 0, \text{ stable}$$

3. Quantity demanded in a market is given by

$$q^d = D(p); \qquad D'(p) < 0$$

and quantity supplied is given by

$$q^s = S(p); \qquad S'(p) \geq 0.$$

If price adjusts according to $\dot{p} = \alpha(D(p) - S(p))$, where $\alpha > 0$ is a constant, conduct a qualitative analysis of the dynamics of market price.

The steady-state value is defined by

$$D(\bar{p}) = S(\bar{p}).$$

Moreover,

$$\left.\frac{d\dot{p}}{dp}\right|_{p=\bar{p}} = \alpha\left(D'(\bar{p}) - S'(\bar{p})\right) < 0$$

if and only if

$$D'(\bar{p}) - S'(\bar{p}) < 0 \Leftrightarrow D'(\bar{p}) < S'(\bar{p}).$$

4. Let y be the proportion of a population that has already been infected with a highly contagious flu as of time t. Let z (a constant) be the fraction of the population that is either innoculated against the flu or is otherwise immune to the flu. Thus $0 \leq y \leq 1 - z$. The rate of change of y is proportional to 1) y (the more people with the flu the faster it spreads) and 2) $1 - y - z$, the proportion that are vulnerable to contracting the flu. The differential equation is

$$\dot{y} = ay(1 - y - z) = a(1 - z)y - ay^2.$$

Conduct a qualitative analysis of the dynamics of flu infection.

The steady state value is the solution to

$$ay(1 - y - z) = 0$$

i.e.,

$$\bar{y}_1, \bar{y}_2 = (0, 1 - z).$$

$$\frac{d\dot{y}}{dy} = a(1 - z) - 2ay.$$

Hence,

$$\left.\frac{d\dot{y}}{dy}\right|_{y=\bar{y}} = a(1 - z) - 2ay|_{y=\bar{y}} \begin{cases} a(1 - z) > 0 & \text{if } \bar{y} = 0 \\ -a(1 - z) < 0 & \text{if } \bar{y} = 1 - z \end{cases}$$

The stable steady state is $\bar{y} = 1 - z$.

5. (Exercise 6, page 887) Let y be the *stock* of carbon dioxide in the atmosphere. Let $x > 0$ (a constant) be the *flow* of carbon dioxide emissions that arise from industrial activity. Assume that the dynamics of y are given by

$$\dot{y} = x - y^a, \qquad x > 0.$$

where the term y^a measures the earth's capacity to remove carbon dioxide from the atmosphere (and absorb it into plant life and oceans). Conduct a qualitative analysis of this model, first for the case that $a > 0$ and then for the case $a < 0$. Comment on your results.

Now steady states are given by

$$x - y^a = 0, \qquad x = y^a, \qquad \bar{y} = x^{1/a}.$$

$$\left.\frac{d\dot{y}}{dy}\right|_{y=\bar{y}} = -ay^{a-1}\big|_{y=\bar{y}} = -ax^{\frac{a-1}{a}} \begin{cases} < 0 & \text{if } a > 0 \\ > 0 & \text{if } a < 0 \end{cases}$$

In the former case, the steady state is stable and carbon dioxide will settle down at $\bar{y} = x^{1/a}$. This can be made lower only by reducing industrial activity. In the latter case, the steady state is unstable. For any $y_0 > x^{1/a}$ carbon dioxide will increase without bound. Since $a < 0$ the larger is the value of x the smaller will be the critical value, $x^{1/a}$.

6. (Exercise 1, page 893) Solve

$$\dot{y} + 2y = \frac{3}{y}.$$

Multiply by y to get

$$y\dot{y} + 2y^2 = 3.$$

Let $x = y^2$. Then $\dot{x} = 2y\dot{y}$.

$$\frac{1}{2}\dot{x} + 2x = 3 \text{ or } \dot{x} + 4x = 6.$$

By the usual method,

$$x(t) = Ce^{-4t} + \frac{3}{2}.$$

Now,

$$x_0 = C + \frac{3}{2}; C = x_0 - \frac{3}{2}.$$

Since $x_0 = y_0^2$ we should insist that $x_0 \geq 0$. Then $C \geq -\frac{3}{2}$ and

$$y(t) = \left(Ce^{-4t} + \frac{3}{2}\right)^{1/2} = \left(\left(y_0^2 - \frac{3}{2}\right)e^{-4t} + \frac{3}{2}\right)^{1/2}$$

7. Find the explicit solution to the flu infection problem.

For an explicit solution write the equation as

$$\dot{y} = a(1 - z)y - ay^2$$

Or

$$y^{-2}\dot{y} = a(1-z)y^{-1} - a$$

Let $x = y^{-1}$. Hence we must have $y > 0$ or at least $y_0 > 0$. Then $\dot{x} = -y^{-2}\dot{y}$.

$$\begin{aligned} -\dot{x} &= a(1-z)x - a \\ \dot{x} &= -a(1-z)x + a \end{aligned}$$

Steady state value is $x = 1/(1-z)$. General solution is

$$x = Ce^{-a(1-z)t} + \frac{1}{1-z}.$$

Definite solution is

$$x = \left(x_0 - \frac{1}{1-z}\right)e^{-a(1-z)t} + \frac{1}{1-z}.$$

This converges to $\dfrac{1}{1-z}$. The general solution for y is

$$y = x^{-1} = \frac{1}{Ce^{-a(1-z)t} + \dfrac{1}{1-z}}$$

This converges to $1-z$.

$$y_0 = \frac{1}{C + \dfrac{1}{1-z}}$$

$$\begin{aligned} y_0\left(C + \frac{1}{1-z}\right) &= 1 \\ y_0\left[(1-z)C + 1\right] &= 1-z \\ C &= \frac{1-z-y_0}{y_0(1-z)} \end{aligned}$$

$$y = \frac{1}{\left(\dfrac{1-z-y_0}{y_0(1-z)}\right)e^{-a(1-z)t} + \dfrac{1}{1-z}}$$

8. Solve

$$\dot{y} = \frac{t^2}{y(1+t^3)}$$

$$y\,dy = \frac{t^2}{1+t^3}\,dt = \frac{1}{3}\frac{3t^2}{1+t^3}\,dt$$

$$\begin{aligned}\frac{y^2}{2} &= \frac{1}{3}\ln(1+t^3) + C' \\ y(t) &= \left(\frac{2}{3}\ln(1+t^3) + C\right)^{1/2}\end{aligned}$$

$$y_0 = C^{1/2}, C = y_0^2.$$

$$y(t) = \left(\frac{2}{3}\ln(1+t^3) + y_0^2\right)^{1/2}$$

Let's check this one.

$$\dot{y} = \frac{1}{2}\left(\frac{2}{3}\ln(1+t^3) + y_0^2\right)^{-1/2}\frac{2t^2}{1+t^3} = \frac{1}{y}\frac{t^2}{1+t^3}\checkmark$$

9. The slope of a consumer's indifference curve is given by

$$\left.\frac{dx_2}{dx_1}\right|_{u\,=\,\text{constant}} = -\frac{\partial u(x_1,x_2)/\partial x_1}{\partial u(x_1,x_2)/\partial x_2}.$$

Suppose

$$\left.\frac{dx_2}{dx_1}\right|_{u\,=\,\text{constant}} = -\frac{\alpha x_1^{\alpha-1}x_2^{\beta}}{\beta x_1^{\alpha}x_2^{\beta-1}}.$$

This is a nonlinear, first-order differential equation. Solve to find the utility function. (Hint: First simplify the RHS. The result will be a differential equation that is separable.)

$$\frac{dx_2}{dx_1} = -\frac{\alpha}{\beta}\frac{x_2}{x_1}.$$

$$\alpha\frac{dx_1}{x_1} + \beta\frac{dx_2}{x_2} = 0$$

$$\alpha\ln x_1 + \beta\ln x_2 = \ln u$$

$$u = x_1^{\alpha}x_2^{\beta}.$$