Economics 511 Problem Set 23
Linear Second-Order Differential Equations
Answers

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1. Find the general solutions to the following differential equations.

(a) $\ddot{y} + 5\dot{y} - 6y = 36$

$$\begin{aligned} r^2 + 5r - 6 &= 0 \\ (r+6)(r-1) &= 0 \end{aligned}$$

$$\begin{aligned} r_1, r_2 &= \frac{-5 \pm \sqrt{25+24}}{2} \\ &= \frac{-5+7}{2}, \frac{-5-7}{2} \\ &= 1, -6. \end{aligned}$$

$$y_h = C_1 e^{-6t} + C_2 e^t$$

$$\frac{b}{a_2} = -6$$

$$y(t) = C_1 e^{-6t} + C_2 e^t - 6$$

$$\begin{aligned} \dot{y} &= -6C_1 e^{-6t} + C_2 e^t \\ \ddot{y} &= 36C_1 e^{-6t} + C_2 e^t \end{aligned}$$

$$\begin{aligned} & \ddot{y} + 5\dot{y} - 6y \\ = \; & 36C_1 e^{-6t} + C_2 e^t + 5\left(-6C_1 e^{-6t} + C_2 e^t\right) - 6\left(C_1 e^{-6t} + C_2 e^t - 6\right) \\ = \; & (36 - 30 - 6)\, C_1 e^{-6t} + (1 + 5 - 6)\, C_2 e^t + 36 = 36 \quad \checkmark \end{aligned}$$

(b) $\ddot{y} + 2\dot{y} + 10y = 30$

$$\begin{aligned} r^2 + 2r + 10 &= 0 \\ r_1, r_2 &= \frac{-2 \pm \sqrt{4-40}}{2} = \frac{-2 \pm 6i}{2} \\ &= -1 \pm 3i, h = -1, v = 3 \end{aligned}$$

$$y(t) = e^{-t}\left(A_1 \cos(3t) + A_2 \sin(3t)\right) + 3$$

$$
\begin{aligned}
\dot{y} &= -e^{-t}\left(A_1\cos(3t) + A_2\sin(3t)\right) + e^{-t}\left(-3A_1\sin(3t) + 3A_2\cos(3t)\right) \\
&= e^{-t}\left[(3A_2 - A_1)\cos(3t) + (-3A_1 - A_2)\sin(3t)\right] \\
\ddot{y} &= -e^{-t}\left[(3A_2 - A_1)\cos(3t) + (-3A_1 - A_2)\sin(3t)\right] \\
&\quad +e^{-t}\left[(3A_1 - 9A_2)\sin(3t) + (-9A_1 - 3A_2)\cos(3t)\right] \\
&= e^{-t}\left[(-8A_1 - 6A_2)\cos(3t) + (6A_1 - 8A_2)\sin(3t)\right]
\end{aligned}
$$

$$
\begin{aligned}
& \ddot{y} + 2\dot{y} + 10y \\
= \; & e^{-t}\left[(-8A_1 - 6A_2)\cos(3t) + (6A_1 - 8A_2)\sin(3t)\right] \\
& +2e^{-t}\left[(3A_2 - A_1)\cos(3t) + (-3A_1 - A_2)\sin(3t)\right] \\
& +10\left[e^{-t}\left(A_1\cos(3t) + A_2\sin(3t)\right) + 3\right] \\
= \; & 30 \quad \checkmark
\end{aligned}
$$

(c) $\ddot{y} - 4\dot{y} + 4y = 4$

$$
\begin{aligned}
r^2 - 4r + 4 &= 0 \\
(r-2)^2 &= 0
\end{aligned}
$$

$$
y(t) = C_1e^{2t} + C_2te^{2t} + 1
$$

$$
\begin{aligned}
\dot{y} &= 2C_1e^{2t} + 2C_2te^{2t} + C_2e^{2t} \\
\ddot{y} &= 4C_1e^{2t} + 4C_2te^{2t} + 2C_2e^{2t} + 2C_2e^{2t} \\
&= 4C_1e^{2t} + 4C_2te^{2t} + 4C_2e^{2t} \\
\ddot{y} - 4\dot{y} + 4y &= 4C_1e^{2t} + 4C_2te^{2t} + 4C_2e^{2t} \\
&\quad -\left(8C_1e^{2t} + 8C_2te^{2t} + 4C_2e^{2t}\right) \\
&\quad +4C_1e^{2t} + 4C_2te^{2t} + 4 \\
&= 4 \quad \checkmark
\end{aligned}
$$

(d) $\ddot{y} - 3\dot{y} = 12$

$$
\begin{aligned}
r^2 - 3r &= 0 \\
r(r-3) &= 0 \\
r_1, r_2 &= (0, 3)
\end{aligned}
$$

$$
y_h = C_1 + C_2e^{3t}
$$

$$
\begin{aligned}
y_p &= A_1t \\
\dot{y}_p &= A_1
\end{aligned}
$$

$$
-3A_1 = 12; A_1 = -4
$$

$$
\begin{aligned}
y(t) &= C_1 + C_2 e^{3t} - 4t \\
\dot{y} &= 3C_2 e^{3t} - 4 \\
\ddot{y} &= 9C_2 e^{3t} \\
9C_2 e^{3t} - 3\left(3C_2 e^{3t} - 4\right) &= 12 \quad \checkmark
\end{aligned}
$$

2. For parts (a) - (c) in question 1 determine the constants of integration given the following initial values.

   (a) $y(0) = 11, y'(0) = 3$

$$
\begin{aligned}
y(t) &= C_1 e^{-6t} + C_2 e^{t} - 6 \\
y'(t) &= -6C_1 e^{-6t} + C_2 e^{t}
\end{aligned}
$$

$$
\begin{aligned}
y(0) &= 11 = C_1 + C_2 - 6 \\
y'(0) &= 3 = -6C_1 + C_2
\end{aligned}
$$

$$C_1 = 2, C_2 = 15$$

   (b) $y(0) = 6, y'(0) = 9$

$$
\begin{aligned}
y(t) &= e^{-t}\left(A_1 \cos(3t) + A_2 \sin(3t)\right) + 3 \\
y'(t) &= -e^{-t}\left(A_1 \cos(3t) + A_2 \sin(3t)\right) \\
&\quad + e^{-t}\left(-3A_1 \sin(3t)\right) + 3A_2 \cos(3t)
\end{aligned}
$$

$$
\begin{aligned}
y(0) &= 6 = A_1 + 3 \\
y'(0) &= 9 = -A_1 + 3A_2
\end{aligned}
$$

$$A_1 = 3, A_2 = 4$$

   (c) $y(0) = 2, y'(0) = 4$

$$
\begin{aligned}
y(t) &= C_1 e^{2t} + C_2 t e^{2t} + 1 \\
y'(t) &= 2C_1 e^{2t} + 2C_2 t e^{2t} + C_2 e^{2t}
\end{aligned}
$$

$$
\begin{aligned}
y(0) &= 2 = C_1 + 1 \\
y'(0) &= 4 = 2C_1 + C_2
\end{aligned}
$$

$$C_1 = 1, C_2 = 2$$

3. Provide three examples of linear, second-order differential equations with constant coefficients that result in 1) distinct real roots 2) repeated real roots and 3) complex roots. Then solve them.

   A linear, second-order differential equation with constant coefficients has the general form

$$\ddot{y} + a_1\dot{y} + a_2 y = b$$

   The roots of the characteristic equation

$$r^2 + a_1 r + a_2 = 0$$

   are given by

$$r_1, r_2 = \frac{-a_1 \pm \sqrt{a_1^2 - 4a_2}}{2}$$

   For distinct real roots choose a_1 and a_2 so that $a_1^2 - 4a_2 > 0$.

   For repeated real roots choose a_1 and a_2 so that $a_1^2 - 4a_2 = 0$.

   For complex roots a_1 and a_2 so that $a_1^2 - 4a_2 < 0$.