Economics 511 Problem Set 25
Optimal Control Theory

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1. (Section 25.1, Exercise 1, page 1012) Solve

$$\begin{aligned} \max J &= \int_0^T -\left(ay + by^2\right)\, dt, \qquad a, b, > 0 \\ \text{subject to } \dot{x} &= x - y \\ x(0) &= x_0 \end{aligned}$$

   Easy to solve differential equations.

2. (Section 25.1, Exercise 5, page 1012) Solve

$$\begin{aligned} \max J &= \int_0^T \left(y - y^2 - 4x - 3x^2\right)\, dt \\ \text{subject to } \dot{x} &= x + y \\ x(0) &= x_0 \end{aligned}$$

   A chance to solve a system of linear differential equations.

3. (Section 25.2, Exercise 3, page 1025) Solve

$$\begin{aligned} \max J &= \int_0^T e^{-\rho t}\left(yx - y^2 - x^2\right)\, dt \\ \text{subject to } \dot{x} &= x + y \\ x(0) &= x_0 \end{aligned}$$

   Learn about current-valued Hamiltonians and current-valued costate variables.

4. (Section 25.2, Exercise 7, page 1025) Conduct a qualitative analysis using a phase diagram [in (I, K) space] of the following investment problem:

$$\begin{aligned} \max J &= \int_0^T e^{-\rho t}\left[K - aK^2 - c(I)\right]\, dt \\ \text{subject to } \dot{K} &= I - \delta K \\ K(0) &= K_0 \end{aligned}$$

   where $c(I)$ is the investment cost function with the following properties: $c'(I) > 0, c''(I) > 0$ and $c(0) = 0$.

   Phase diagrams: indispensable method in the dynamic macro toolkit.

5. (Section 25.3, Exercise 1, page 1038) Solve the following:

$$\begin{aligned} \max J &= \int_0^T e^{-\rho t}\left(yx - y^2 - x^2\right)\, dt \\ \text{subject to } \dot{x} &= x + y \\ x(0) &= x_0 \\ x(T) &= x_T \end{aligned}$$

   A boundary condition variation on Problem 3 above.

6. (Section 25.3, Exercise 5, page 1039) The production of a good, y, yields economic benefits but also contributes to the stock of pollution, x, which is an economic bad. Instantaneous net benefits are $y - y^2 - x^2$. The stock of pollution depreciates (is broken down naturally in the environment) at the rate δ. The optimal path of consumption, y, is the solution to

$$\begin{aligned} \max J &= \int_0^T e^{-\rho t}\left(y - y^2 - x^2\right)\, dt \\ \text{subject to } \dot{x} &= y - \delta x \\ x(0) &= x_0 \\ x(T) &= x_T \end{aligned}$$

   The current-valued Hamiltonian is

$$\mathcal{H} = y - y^2 - x^2 + \mu\left(y - \delta x\right)$$

$$\begin{aligned} \frac{\partial \mathcal{H}}{\partial y} &= 1 - 2y + \mu = 0 \\ y &= \frac{1+\mu}{2}. \end{aligned}$$

   For the current-valued costate variable,

$$\dot{\mu} - \rho\mu = -\frac{\partial \mathcal{H}}{\partial x} = 2x + \mu\delta$$

   Also,

$$\dot{x} = y - \delta x = \frac{1+\mu}{2} - \delta x$$

   and thus the system is

$$\begin{aligned} \dot{\mu} &= \left(\rho + \delta\right)\mu + 2x \\ \dot{x} &= \frac{1+\mu}{2} - \delta x. \end{aligned}$$

Draw the phase diagram. Suppose that $x_0 < x_T < \bar{x}$, where $\bar{x}$ is the steady-state value of x. Show that it is possible that if T is not large enough, a solution does not exist. Also show that as T becomes very large, the solution path approaches the saddle path.

7. (Exercise 1, page 1052) Solve the following free-endpoint, infinite-time horizon consumption model for the optimal consumption path, $c(t)$. For this problem, assume that $\rho - r < 0$.

$$\max J = \int_0^\infty e^{-\rho t} \left[c - ac^2\right] \, dt$$

$$\text{subject to} \quad \dot{x} = rx - c$$

$$x(0) = x_0$$

$$x(t) \geq 0$$

8. (Exercise 5, page 1052) Solve the following exhaustible resource problem for the optimal path of extraction, $y(t)$.

$$\max J = \int_0^\infty e^{-\rho t} \ln y \, dt$$

$$\text{subject to} \quad \dot{x} = -y$$

$$x(0) = x_0$$

$$x(t) \geq 0$$

9. (Exercise 1, page 1062) Assume that $\rho < r$ and solve the following linear optimal consumption model. Use a phase diagram to assist.

$$\max J = \int_0^T e^{-\rho t} c \, dt$$

$$\text{subject to} \quad \dot{x} = rx - c$$

$$x(0) = x_0$$

$$x(T) = x_T$$

$$0 \leq c \leq c_{\max}$$