

Here are some different characterizations of the natural number  $e$ .

Suppose  $f(x) = a^x$ . Find a value of  $a$  such that

$$\frac{d}{dx}a^x = a^x.$$

In general

$$\frac{d}{dx}a^x = \lim_{h \rightarrow 0} \left[ \frac{a^{x+h} - a^x}{h} \right] = \lim_{h \rightarrow 0} \left[ \frac{a^h - 1}{h} \right] a^x$$

So the goal is to find a value of  $a$  such that

$$\lim_{h \rightarrow 0} \left[ \frac{a^h - 1}{h} \right] = 1$$

For small  $h$

$$\frac{a^h - 1}{h} \simeq 1$$

or

$$a \simeq (h + 1)^{\frac{1}{h}}.$$

We conclude that

$$a = \lim_{h \rightarrow 0} (h + 1)^{\frac{1}{h}}$$

Call the answer  $e$  and let  $x = \frac{1}{h}$ . Then

$$e = \lim_{x \rightarrow \infty} \left( 1 + \frac{1}{x} \right)^x.$$

## Continuous Compounding

Suppose you put one dollar in the bank and earn an annual interest rate of  $r \times 100\%$ . After one year you will have  $1 + r$  dollars. After  $n$  years you will have  $(1 + r)^n$  dollars. Example: If  $r = .10$  then  $1 + r = 1.10$  and if  $n = 10$  then  $(1 + 0.1)^{10} = 2.5937$ . Now suppose the interest is compounded  $m$  times per year at the rate  $r/m$ . At the end of a year you will have the future value

$$\left( 1 + \frac{r}{m} \right)^m$$

For example,

$$\left(1 + \frac{0.1}{12}\right)^{12} = 1.1047$$

After  $t$  years you will have

$$\left(1 + \frac{r}{m}\right)^{mt}$$

Now rewrite this as

$$\begin{aligned} k(t) &= \left(1 + \frac{r}{m}\right)^{mt} \\ &= \left(1 + \frac{1}{m/r}\right)^{r(m/r)t} \\ &= \left[\left(1 + \frac{1}{x}\right)^x\right]^{rt}, \quad x = m/r. \end{aligned}$$

Then

$$\lim_{x \rightarrow \infty} k(t) = \lim_{x \rightarrow \infty} \left[\left(1 + \frac{1}{x}\right)^x\right]^{rt} = e^{rt}.$$

An initial amount  $k(0)$  invested today will have a future value

$$k(t) = k(0)e^{rt} \tag{1}$$

in  $t$  years.

There is another interesting way to get this result. With discrete time the value of an asset grows according to the formula

$$k(t) = (1 + r)^t k(0)$$

If we extend the time period from  $t$  to  $t + \Delta t$  then

$$k(t + \Delta t) = k(t) + rk(t)\Delta t.$$

where the interest is applied to the asset value at time  $t$  multiplied by the fraction of a year,  $\Delta t$ . (So  $\Delta t = 1/m$ ). Rearranging,

$$\frac{k(t + \Delta t) - k(t)}{\Delta t} = rk(t)$$

In the limit, as  $\Delta t \rightarrow 0$  we get

$$\lim_{\Delta t \rightarrow 0} \frac{k(t + \Delta t) - k(t)}{\Delta t} = \frac{dk(t)}{dt} = rk(t)$$

or

$$\frac{1}{k(t)} \frac{dk(t)}{dt} = r.$$

The lefthand side is the derivative of  $\ln k(t)$ . Thus, integrating

$$\ln k(t) = rt + \ln c$$

and

$$k(t) = ce^{rt}$$

Now, since  $k(0) = ce^0 = c$  we have

$$k(t) = k(0)e^{rt}. \tag{2}$$

### Present Value in Continuous Time

If the interest rate is  $r$  then the present value of one dollar received one year from now is

$$PV_1 = \frac{1}{1+r} = (1+r)^{-1}$$

since  $PV_1$  dollars invested today would yield  $PV_1(1+r) = 1$  a year from now. If the interest rate,  $r/m$  is earned  $m$  times a year for  $t$  years then

$$\begin{aligned} PV_t &= \left[ \left( 1 + \frac{r}{m} \right)^{mt} \right]^{-1} \\ &= \left[ \left( 1 + \frac{1}{m/r} \right)^{r(m/r)t} \right]^{-1} \\ &= \left[ \left( \left( 1 + \frac{1}{m/r} \right)^{m/r} \right)^{rt} \right]^{-1} \\ &= \left[ \left( \left( 1 + \frac{1}{x} \right)^x \right)^{rt} \right]^{-1}, \quad x = m/r. \end{aligned}$$

and

$$\lim_{x \rightarrow \infty} PV_t = \lim_{x \rightarrow \infty} \left[ \left( 1 + \frac{1}{x} \right)^x \right]^{-rt} = e^{-rt}.$$

An amount  $A$  received  $t$  years from now has a present value given by

$$PV_t = Ae^{-rt}.$$

Of course we could have just rearranged (2) to get

$$k(0) = k(t)e^{-rt}.$$