Economics 540B Final Exam
Suggested Answers

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Here are some suggested answers to the final exam in ECON 540B. If you find any errors or omissions please report them to me and I will revise this document accordingly. - DP.

1. When inputs prices are $(w_1, w_2) = (1, 1)$, the Hessian of the cost function is given by

$$\begin{bmatrix} \dfrac{\partial^2 c(w,y)}{\partial w_1^2} & \dfrac{\partial^2 c(w,y)}{\partial w_1 \partial w_2} \\ \dfrac{\partial^2 c(w,y)}{\partial w_2 \partial w_1} & \dfrac{\partial^2 c(w,y)}{\partial w_2^2} \end{bmatrix} = \begin{bmatrix} a & b \\ 1/2 & c \end{bmatrix}.$$

What restrictions, if any, should be imposed on the parameters a, b, and c?

This Hessian matrix is symmetric and negative semidefinite. Symmetry implies that $b = 1/2$. It is also important to note that the cost function is homogeneous of degree one. Hence, each first partial derivative is homogeneous of degree zero. Application of Euler's theorem to the first partial derivatives gives us

$$\frac{\partial^2 c(w,y)}{\partial w_1^2} w_1 + \frac{\partial^2 c(w,y)}{\partial w_1 \partial w_2} w_2 = 0 \tag{1}$$

and

$$\frac{\partial^2 c(w,y)}{\partial w_2 \partial w_1} w_1 + \frac{\partial^2 c(w,y)}{\partial w_2^2} w_2 = 0 \tag{2}$$

Since $(w_1, w_2) = (1, 1)$, equation (1) implies that

$$\begin{aligned} a + b &= 0 \\ &\Rightarrow a = -b = -1/2, \end{aligned}$$

and equation (2) implies that

$$\begin{aligned} \frac{1}{2} + c &= 0 \\ &\Rightarrow c = -1/2. \end{aligned}$$

These results are consistent with negative semidefiniteness.

2. State the Weak Axiom of Cost Minimization (WACM).

   Given a data set of observations of input prices, inputs, and scalar outputs, $(\mathbf{w}^t, \mathbf{x}^t, y^t), t = 1, ..., T$, the WACM is:

   $$\text{WACM} : \mathbf{w}^t\mathbf{x}^t \leq \mathbf{w}^t\mathbf{x}^s \text{ for all } s \text{ and } t \text{ such that } y^t \leq y^s.$$

   (a) Give an example of a simple data set that violates the WACM.

Obs.	w_1	w_2	x_1	x_2	y
1	1	2	2	8	40
2	2	1	5	5	50

   Since $y^1 = 40 \leq 50 = y^2$, the WACM requires that $\mathbf{w}^1\mathbf{x}^1 \leq \mathbf{w}^1\mathbf{x}^2$. However, in this example

   $$\begin{aligned} \mathbf{w}^1\mathbf{x}^1 &= (1)(2) + (2)(8) = 18 \\ &> 15 = (1)(5) + (2)(5) \\ &= \mathbf{w}^1\mathbf{x}^2, \end{aligned}$$

   and so the WACM is violated. By the way, I generated the input-output data from the production function $y = 10\sqrt{x_1 x_2}$.

   (b) Why might a real data set violate the WACM?

   Here are six reasons. They probably do not exhaust all of the possibilities.

   i. The logic behind the WACM is based on the premise that the firm is a price-taker in all of its input markets. If this is not so, i.e., if the firm can influence one or more input prices, then the WACM may fail.

   ii. The logic behind the WACM is also based on the premise that all inputs are variable, i.e., the firm operates in the long run. However, if the data reflects decisions made in the short run (i.e., when some inputs are fixed) then the WACM may fail.

   iii. The logic behind the WACM is also based on the premise that the firm minimizes cost. However, if the managers who run the firm are not the owners of the firm they may expend resources on some inputs in an inefficient way (fancy corner offices adorned with rare art, luxury corporate jets, lavish vacations in Tahiti, etc.). Thus the firm is not minimizing cost and the WACM may be violated. (This is an example of the principal-agent problem.)

   iv. The logic behind the WACM is also based on the premise that there is no uncertainty about input prices or the production function. If there is uncertainty then the WACM may fail.

v. The input prices observed by economists might not be the true shadow prices faced by the firm because of taxes, hidden discount prices, or aggregation errors that occur when there is variation in input prices across firms and only average prices are reported. Again, the WACM may fail.

vi. If the technology has changed from one observation to the next then the WACM may fail.

3. (50) True/False - Justify. Indicate whether each of the following statements is true or false and then justify your answer. Points will be awarded according the quality of your justifications only.

   (a) A firm that operates two plants should produce all of its output in the plant with the lowest average cost.

   FALSE: A firm that wants to minimize total cost wants to solve the problem given by:

   $$\min C^1(y_1) + C^2(y_2) \text{ subject to } y_1 + y_2 \geq y,$$

   where y_i and $C^i(y_i)$ are the output and the cost function for plant i, $i = 1, 2$. If both cost functions are convex functions then this problem can be solved with the Kuhn-Tucker conditions

   $$\begin{aligned} \frac{dC^1(y_1)}{dy_1} - \lambda &\geq 0, \quad y_1 \geq 0, \quad \left(\frac{dC^1(y_1)}{dy_1} - \lambda\right) y_1 = 0 \\ \frac{dC^2(y_2)}{dy_2} - \lambda &\geq 0 \quad y_2 \geq 0, \quad \left(\frac{dC^2(y_2)}{dy_2} - \lambda\right) y_2 = 0 \\ y - y_1 - y_2 &\leq 0 \quad \lambda \geq 0, \quad (y - y_1 - y_2)\lambda = 0 \end{aligned}$$

   If this problem has an interior solution for (y_1, y_2) then output is allocated to equate marginal cost in each plant. On the other hand we might have a solution for which

   $$\begin{aligned} \frac{dC^1(y_1^*)}{dy_1} - \lambda^* &= 0, \quad y_1^* > 0 \\ \frac{dC^2(y_2^*)}{dy_2} - \lambda^* &> 0, \quad y_2^* = 0 \end{aligned}$$

   and, in this case,

   $$\frac{dC^1(y_1^*)}{dy_1} < \frac{dC^2(y_2^*)}{dy_2},$$

   i.e., all of the output is produced in the plant with the lower marginal cost.(See Exercise 4.3 in Varian for an example of a solution that, for some values of y, might involve setting one plant's output to zero.) For an example for which the plant cost functions are not convex see Exercise 4.6, page 63 in Varian.

(b) Given a firm's cost function one can recover the firm's input requirement sets and these sets will be convex and monotonic.

FALSE: Given a cost function one *can* define the following sets:

$$V^*(y) = \{x : c(w, y) \leqq wx \text{ for all } w \geq 0\} . y > 0.$$

It is easy to prove that $V^*(y)$ is convex and monotonic. But, in general, the firm's input requirement sets, $V(y)$, need only be subsets of the $V^*(y), y > 0$. Therefore the statement is false in general. However, if each $V(y)$ is convex and monotonic then $V(y) = V^*(y), y > 0$. In this case one can recover $V(y)$ by computing $V^*(y)$ and the resulting input sets are convex and monotonic.

(c) A given data set consists of T observations of a consumer's consumption choices at different prices, i.e., $(p^t, x^t), t = 1, 2, ..., T$, and it satisfies the following inequalities

$$p^t \cdot x^t < p^t \cdot x^s, s, t = 1, 2, ..., T, s \neq t.$$

Therefore this data set satisfies the Generalized Axiom of Revealed Preference.

TRUE: The GARP is given by: If

$$x^t R x^s \qquad (p^t x^t \geq p^t x^j, p^j x^j \geq p^j x^k, ..., p^n x^n \geq p^n x^s)$$

then it is *not* the case that $x^s P^D x^t$(it is the case that $p^s x^s \leq p^s x^t$). Since

$$p^t \cdot x^t < p^t \cdot x^s, s, t = 1, 2, ..., T, s \neq t.$$

we have the result that $\text{not}(x^t R x^s), s, t = 1, 2, ..., T, s \neq t$. Thus the GARP is never violated. Moreover, we always have $\text{not}\big(x^s P^D x^t\big), s, t = 1, 2, ..., T, s \neq t$, and again the GARP is never violated.

(d) If a consumer's utility function is given by $u(x_1, x_2) = \ln x_1 + x_2$ then her Marshallian demand for good one is independent of income.

TRUE: From the first-order conditions

$$\begin{aligned} \frac{\partial u(x_1, x_2)}{\partial x_1} - \lambda p_1 &= \frac{1}{x_1} - \lambda p_1 = 0 \\ \frac{\partial u(x_1, x_2)}{\partial x_2} - \lambda p_2 &= 1 - \lambda p_2 = 0 \end{aligned}$$

we get

$$x_1^* = \frac{1}{\lambda p_1} = \frac{p_2}{p_1},$$

and so her Marshallian demand for good one is independent of income.

(e) Stan's utility function is $u(x_1, x_2) = \min\{x_1, x_2\}$. Oliver's utility function is $v(x_1, x_2) = \min\{x_1^2, x_2\}$. Stan and Oliver have the same preference ordering.

FALSE:

$$u(4,5) = \min\{4,5\} = 4 > 3 = \min\{3,6\} = u(3,6)$$

$$v(4,5) = \min\left\{4^2, 5\right\} = 5 < 6 = \min\left\{3^2, 6\right\} = v(3,6).$$

Stan prefers $(4,5)$ to $(3,6)$ while Oliver prefers $(3,6)$ to $(4,5)$. They do not have the same preference ordering.

(f) When Susan is faced with two alternatives:

G_1: A 100 percent chance of receiving \$10,000.

G_2: A 50 percent chance of receiving \$20,000, a 10 percent chance of receiving \$10,000, and 40 percent chance of receiving nothing.

she prefers G_1 to G_2.

True or false: "If Susan's preferences satisfy the expected utility axioms then she prefers G_3 to G_4 where G_3 and G_4 are given by

G_3: A 90 percent chance of receiving \$10,000 and 10 percent chance of receiving nothing.

G_4: A 50 percent chance of receiving \$20,000 and a 50 percent chance of receiving nothing."

TRUE: Since she prefers G_1 to G_2

$$u(10K) > 0.5u(20K) + 0.10u(10K) + 0.40u(0),$$

where $u(\cdot)$ is Susan's utility function. Subtract $0.10u(10K)$ from both sides and add $0.10u(0)$ to both sides to get

$$0.90u(10K) + 0.10u(0) > 0.5u(20K) + 0.50u(0),$$

and hence she prefers G_3 to G_4.

4. (15) Your friend, who lives in a two-good world, claims that his Marshallian demands are given by

$$x_i = a_i + b_i\left(\frac{m}{p_i}\right), i = 1, 2.$$

Of course, there is a list of conditions that all Marshallian demand functions must satisfy. State each of these conditions and impose them, *one by one*, on the above demand system to derive the restrictions on the demand parameters implied by each condition. Show all of your work.

The Marshallian demands are clearly **homogeneous of degree zero** in (p, m). This condition imposes no restrictions on the parameters. The **adding-up** condition requires

$$\sum p_i x_i = \sum p_i \left[a_i + b_i \left(\frac{m}{p_i} \right) \right] = a_1 p_1 + a_2 p_2 + (b_1 + b_2) m \equiv m. \tag{3}$$

Note that the right-hand side of (3) does not depend on p_1 or p_2. This requires that

$$a_1 = 0 \text{ and } a_2 = 0. \tag{4}$$

(To see this, differentiate both sides of (3) with respect to p_1. We get $a_1 = 0$. Similarly, $a_2 = 0$.)

Now (3) becomes

$$(b_1 + b_2) m \equiv m. \tag{5}$$

Differentiate both sides of (5) with respect to m. We get

$$b_1 + b_2 = 1. \tag{6}$$

If we now impose the conditions given in equations (4) and (6), we get the following demand functions.

$$x_1 = \frac{b_1 m}{p_1} \text{ and } x_2 = \frac{b_2 m}{p_2}, \text{ where } b_1 + b_2 = 1.$$

Let's construct the Slutsky matrix.

$$S = [S_{ij}]_{2\times 2}$$

where

$$S_{ij} = \frac{\partial x_i}{\partial p_j} + \frac{\partial x_i}{\partial m} x_j.$$

Then

$$\begin{aligned} S_{11} &= \frac{\partial x_1}{\partial p_1} + \frac{x_1}{\partial m} x_1 \\ &= -\frac{b_1 m}{p_1^2} + \frac{b_1}{p_1} \left(\frac{b_1 m}{p_1} \right) \\ &= b_1 (b_1 - 1) \frac{m}{p_1^2} \end{aligned}$$

Similarly,

$$S_{22} = b_2 (b_2 - 1) \frac{m}{p_2^2}.$$

Next,

$$\begin{aligned} S_{12} &= 0+\frac{b_1}{p_1}\left(\frac{b_2 m}{p_2}\right) \\ &= \frac{b_1 b_2 m}{p_1 p_2} \end{aligned}$$

and

$$\begin{aligned} S_{21} &= 0+\frac{b_2}{p_2}\left(\frac{b_1 m}{p_1}\right) \\ &= \frac{b_1 b_2 m}{p_1 p_2} \end{aligned}$$

So the Slutsky substitution matrix is

$$S=\begin{bmatrix} b_1\left(b_1-1\right)\left(\dfrac{m}{p_1^2}\right) & \dfrac{b_1 b_2 m}{p_1 p_2} \\ \dfrac{b_1 b_2 m}{p_1 p_2} & b_2\left(b_2-1\right)\left(\dfrac{m}{p_2^2}\right) \end{bmatrix} \tag{7}$$

This Slutsky matrix is **symmetric**, i.e., $S_{12}=S_{21}$. It should also be n**egative semidefinite**.

Negative semidefiniteness. requires that

$$\begin{aligned} S_{11} &= b_1\left(b_1-1\right)\left(\frac{m}{p_1^2}\right)\leq 0 \\ &\Rightarrow b_1\left(b_1-1\right)\leq 0 \end{aligned}$$

This is satisfied in two cases: (1) $b_1=0$ and (2) $b_1>0\Rightarrow b_1-1\leq 0\Rightarrow b_1\leq 1$. Similarly for S_{22} and we get the restrictions: $0\leq b_1\leq 1, 0\leq b_2\leq 1$. Moreover,

$$|S|=b_1\left(b_1-1\right)\left(\frac{m}{p_1^2}\right)b_2\left(b_2-1\right)\left(\frac{m}{p_2^2}\right)-\left(b_1 b_2\frac{m}{p_1 p_2}\right)^2\geq 0.$$

Dividing by $(m^2/p_1^2 p_2^2)$ we get

$$\begin{aligned} b_1\left(b_1-1\right)b_2\left(b_2-1\right)-b_1^2 b_2^2 &= \left(b_1^2-b_1\right)\left(b_2^2-b_2\right)-b_1^2 b_2^2 \\ &= b_1^2 b_2^2-b_1 b_2^2-b_2 b_1^2+b_1 b_2-b_1^2 b_2^2 \\ &= b_1 b_2\left(1-b_1-b_2\right)=0, \end{aligned}$$

where the last equality follows from (6)

The given Marshallian demands satisfy all of the required properties if and only if

$$a_1 = 0, a_2 = 0, \quad 0 \leq b_1 \leq 1, \quad 0 \leq b_2 \leq 1 \text{ and } b_1 + b_2 = 1.$$

Hence,

$$x_1 = \frac{b_1 m}{p_1}, x_2 = \frac{b_2 m}{p_2}, , \quad 0 \leq b_1 \leq 1, \quad 0 \leq b_2 \leq 1 \text{ and } b_1 + b_2 = 1.$$

5. (15) There are two types of consumers in an economy with two consumer goods. There are m consumers of type l with utility functions

$$u_l = x_{l1}^{\alpha} x_{l2}^{1-\alpha}, \qquad l = 1, ..., m,$$

and there are n consumers of type i with utility functions

$$u_i = \min\{x_{i1}, x_{i2}\}, \qquad i = 1, ..., n.$$

Each of the $m + n$ consumers supply one unit of labor to earn wage w.

There are J firms in the economy that produce the two goods according to production functions

$$y_{1j} = aL_{1j} \text{ and } y_{2j} = bL_{2j}, \qquad j = 1, ..., J.$$

(a) Find the market demands for the two goods.

Type ℓ consumers have Cobb-Douglas demands given by

$$x_{\ell 1} = \alpha \frac{w}{p_1} \text{ and } x_{\ell 2} = (1-\alpha)\frac{w}{p_2}.$$

Equilibrium will require that the Cobb-Douglas demands be finite and therefore the equilibrium prices p_1 and p_2 will have to be positive. Type i consumers have Leontief demands given by

$$x_{i1} = x_{i2} = \frac{w}{p_1 + p_2}$$

The market demands are just the sums,

$$\begin{aligned} x_1 &= \sum_{l=1}^{m} x_{l1} + \sum_{i=1}^{n} x_{i1} = \sum_{l=1}^{m} \alpha \frac{w}{p_1} + \sum_{i=1}^{n} \frac{w}{p_1 + p_2} \\ &= m\alpha \frac{w}{p_1} + n \frac{w}{p_1 + p_2} \\ x_2 &= \sum_{l=1}^{m} x_{l2} + \sum_{i=1}^{n} x_{i2} = \sum_{l=1}^{m} (1-\alpha)\frac{w}{p_1} + \sum_{i=1}^{n} \frac{w}{p_1 + p_2} \\ &= m(1-\alpha)\frac{w}{p_2} + n\frac{w}{p_1 + p_2}. \end{aligned}$$

(b) Determine the Walrasian equilibrium price ratios. Determine the total output of each good. Determine how much labor is used to produce x_1 and how much labor is used to produce x_2.

Since all J firms have constant-returns-to-scale technologies they can be treated as one large competitive firm with production functions

$$y_1 = aL_1 \text{ and } y_2 = bL_2.$$

Since all consumers have interior solutions for goods 1 and 2, the firm will produce positive amount of goods 1 and 2. This requires that $L_1 > 0$ and $L_2 > 0$ given the reasonable assumption that $a > 0$ and $b > 0$. Moreover, This large firm will earn zero profits and so

$$\begin{aligned} p_1 y_1 - wL_1 &= p_1 a L_1 - wL_1 \\ &= (p_1 a - w) L_1 = 0 \end{aligned}$$

Since $L_1 > 0$ this implies that

$$\frac{w}{p_1} = a.$$

Similarly,

$$\begin{aligned} p_2 y_2 - wL_2 &= p_2 b L_2 - wL_2 \\ &= (p_2 b - w) L_2 = 0, \end{aligned}$$

and so

$$\frac{w}{p_2} = b.$$

Then

$$\begin{aligned} y_1 &= x_1 = m\alpha \frac{w}{p_1} + n \frac{w}{p_1 + p_2} \\ &= m\alpha \frac{w}{p_1} + n \frac{1}{\frac{p_1}{w} + \frac{p_2}{w}} \\ &= m\alpha a + n \frac{1}{\frac{1}{a} + \frac{1}{b}} \\ x_1 &= m\alpha a + n \frac{ab}{a+b} \end{aligned}$$

Similarly,

$$x_2 = m(1-\alpha)b + n\frac{ab}{a+b}$$

Since

$$\begin{aligned} x_1 &= y_1 = \sum_{j=1}^{J} y_{1j} = a \sum_{j=1}^{J} L_{1j} = aL_1 \\ x_2 &= y_2 = \sum_{j=1}^{J} y_{2j} = b \sum_{j=1}^{J} L_{2j} = bL_2 \end{aligned}$$

we get

$$L_1 = \frac{x_1}{a} = m\alpha + n\frac{b}{a+b}$$

and

$$L_2 = \frac{x_2}{b} = m(1-\alpha) + n\frac{a}{a+b}.$$

It is easy to check that

$$L_1 + L_2 = m + n.$$