Problem Set 1: Technology and Profit — Spring 2009
Answers

1. Consider two netput vectors, $\mathbf{y}$ and $\mathbf{y}'$ in Y. By divisibility, $t\mathbf{y}$ and $(1-t)\mathbf{y}'$ are in Y for all $0 \le t \le 1$. By additivity, $t\mathbf{y} + (1-t)\mathbf{y}'$ is in Y. Thus Y is convex.

   Now suppose $\mathbf{y}$ is in Y. Given any real number, $s > 0$, let n be a nonnegative integer such that $n \ge s \ge n-1$. By additivity, $n\mathbf{y}$ is in Y. Since $s/n \le 1$, then, by divisibility, $(s/n)\, n\mathbf{y} = s\mathbf{y}$ is in Y. Hence, constant returns to scale.

2. $y = f(\mathbf{x}) = Ax_1^a x_2^b$

   (a) $y = f(\mathbf{x}) = Ax_1^a x_2^b$. $y(t) = A\,(tx_1)^a\,(tx_2)^b = t^{a+b}y$.

   $$\left.\frac{dy(t)}{dt}\frac{t}{y}\right|_{t=1} = \left.(a+b)t^{a+b-1}y\frac{t}{y}\right|_{t=1} = a+b.$$

   (b) Add z and define $F(\mathbf{x}, z) = zf(\mathbf{x}/z)$. F is clearly H.D.1 in (x, z). For our example

   $$F(\mathbf{x}, z) = zA\,(x_1/z)^a\,(x_2/z)^b = Ax_1^a x_2^b z^{1-a-b}.$$

   The new input z is the input that, when fixed at $\bar{z}$, gives rise to a short-run production function $y = Bx_1^a x_2^b$, where $B = A\bar{z}^{1-a-b}$. If $a+b < 1$ then z is a "good" input. If $a+b > 1$ then z is a "bad" input.

3. We consider $y = \ln x$.

   (a) This is a bad choice because when $0 < x < 1$, the output is negative.

   (b) $y = \ln(x+1)$ is a better choice since $y \ge 0$ for all $x \ge 0$.

   (c) $y = \ln(x+1) \Rightarrow e^y = x+1$. Let $y_1 = -x$ and let $y_2 = y$. Substituting we get $e^{y_2} = 1 - y_1$ or $y_1 + e^{y_2} - 1 = 0$. Now let

   $$Y = \left\{ y \in R^2 : y_1 + e^{y_2} - 1 \le 0 \right\}.$$

   Note that if $(y_1, y_2) \in Y$ then $y_1 + e^{y_2} - 1 \le 0$ and if $(y_1', y_2') \le (y_1, y_2)$ then $y_1' + e^{y_2'} - 1 \le 0$ i.e., then $(y_1', y_2') \in Y$. Thus Y, as constructed, satisfies netput monotonicity.

4. Solve

   $$\max_{y_1, y_2} p_1 y_1 + p_2 y_2 \text{ subject to } y_1 + y_2^2 \le 0.$$

   Form a Lagrangian

   $$\mathcal{L} = p_1 y_1 + p_2 y_2 + \lambda\left(-y_1 - y_2^2\right).$$

Then

$$\begin{aligned}\frac{\partial \mathcal{L}}{\partial y_1} &= p_1 - \lambda = 0 \\ \frac{\partial \mathcal{L}}{\partial y_2} &= p_2 - 2\lambda y_2 = 0.\end{aligned}$$

We get

$$y_2^* = \frac{1}{2}\frac{p_2}{p_1} \text{ and } y_1 = -\left(y_2^*\right)^2 = -\frac{1}{4}\left(\frac{p_2}{p_1}\right)^2.$$

Hence,

$$\begin{aligned}\pi(p_1, p_2) &= p_1 y_1^* + p_2 y_2^* \\ &= -\frac{1}{4}\frac{p_2^2}{p_1} + \frac{1}{2}\frac{p_2^2}{p_1} \\ &= \frac{1}{4}\frac{p_2^2}{p_1}.\end{aligned}$$

5.

Period	p_1	p_2	y_1	y_2
1	2	8	-4	2
2	6	12	-1	1

Let $\mathbf{p}^s = (p_1^s, p_2^s)$ and $\mathbf{y}^s = (y_1^s, y_2^s)$ for $s = 1, 2$. Then

$$\begin{aligned}\mathbf{p}^1 &= (2, 8) \quad \mathbf{y}^1 = (-4, 2) \\ \mathbf{p}^2 &= (6, 12) \quad \mathbf{y}^2 = (-1, 1) \\ \mathbf{p}^1\mathbf{y}^1 &= 8 > \mathbf{p}^1\mathbf{y}^2 = 6 \\ \mathbf{p}^2\mathbf{y}^2 &= 6 > \mathbf{p}^2\mathbf{y}^1 = 0.\end{aligned}$$

WAPM is satisfied.

6. Definition of NDRS for the scalar output case: $f(tx) \geq tf(x)$ for all $t \geq 1$. Now if $pf(x) - wx > 0$ for some $x \geq 0$ then

$$pf(tx) - wtx \geq ptf(x) - wtx = t(pf(x) - wx) > pf(x) - wx$$

for all $t > 1$. Hence, profit is unbounded, i.e., $\pi(\mathbf{p}) = +\infty$. Otherwise, $\pi(\mathbf{p}) = 0$. (If $pf(x) - wx < 0$ for all $x \geq 0$ then the firm will shut down and profit will be zero.)

7. WAPM may fail because

   (a) Not all goods are traded in competitive markets so that the firm is not a price taker for those goods. E.g., output monopolist, input monopsonist.

(b) Observed prices may not be the correct shadow prices because of regulation or taxation.

(c) Technical progress has changed the technology set over time.

(d) The firm is managed by someone who did not take ECON 540B.

(e) Principal-agent problem. (Enron)

8. Approximations

(a) Given a data set, $\mathbf{y}^t$ for $t = 1, ..., T$ the inner approximation is

$$\begin{aligned} YI &= \text{convex, monotonic hull of } \left\{\mathbf{y}^t : t = 1, ..., T\right\} \\ &= \left\{\mathbf{y} : \mathbf{y} \leq \sum z_t \mathbf{y}^t, \sum z_t = 1\right\}. \end{aligned}$$

This is valid, i.e., $YI \subseteq Y$, if we assume that Y is convex and monotonic. (Y is monotonic if $\mathbf{y} \in Y$ & $\mathbf{y}' \leq \mathbf{y}$ implies that $\mathbf{y}' \in Y$.) The construction of YI requires only quantity data, i.e., the T observations of the netput vectors.

(b) Given a data set, $(\mathbf{p}^t, \mathbf{y}^t)$ for $t = 1, ..., T$ the outer approximation is

$$YO = \left\{\mathbf{y} : \mathbf{p}^t \mathbf{y} \leq \mathbf{p}^t \mathbf{y}^t \text{ for all } t = 1, ..., T\right\}.$$

This is valid, i.e., $Y \subseteq YO$, if the firm is a competitive profit maximizer. The construction of YO requires both price and quantity data: $(\mathbf{p}^t, \mathbf{y}^t)$ for $t = 1, ..., T$.

9. Given $y = x + x^{1/2}$. Solve

$$\max_x \left\{p\left(x + x^{1/2}\right) - wx\right\} = \max_x \left\{px^{1/2} + (p - w)\, x\right\}.$$

If $p - w \geq 0$ then $px^{1/2} + (p - w)\, x \to +\infty$ as $x \to +\infty$. For a solution, we must have $p - w < 0$ or $w > p$.

$$\begin{aligned} \frac{1}{2} p x^{-1/2} + (p - w) &= 0 \Rightarrow x^{1/2} = \frac{p}{2(w - p)} \Rightarrow x = \frac{p^2}{4(w - p)^2} \\ \pi(p, w) &= p \frac{p}{2(w - p)} + (p - w)\left(\frac{p^2}{4(w - p)^2}\right) \\ &= \frac{p^2}{2(w - p)} - \frac{p^2}{4(w - p)} = \frac{p^2}{4(w - p)}. \end{aligned}$$

Hence,

$$\pi(p, w) = \begin{cases} +\infty & \text{if } w \leq p \\ \dfrac{p^2}{4(w - p)} & \text{if } w > p \end{cases}$$

10. For $x > 1$, we have $y = \ln x$.

$$\begin{aligned} & \max_x \{p \ln x - wx\} \\ \frac{p}{x} - w &= 0 \\ x &= \frac{p}{w}, y = \ln p - \ln w \\ \pi(p, w) &= p \ln\left(\frac{p}{w}\right) - w\left(\frac{p}{w}\right) \\ \pi(p, w) &= p \ln\left(\frac{p}{w}\right) - p.. \end{aligned}$$

When is $p \ln\left(\frac{p}{w}\right) - p \geq 0$? When $\ln\left(\frac{p}{w}\right) \geq 1$, i.e., when $\frac{p}{w} \geq e$. Hence,

$$\pi(p, w) = \begin{cases} 0 & \text{if } \frac{p}{w} < e \\ p \ln\left(\frac{p}{w}\right) - p & \text{if } \frac{p}{w} \geq e \end{cases}$$

11. The quantity of widgets supplied

    (a) will not change

    (b) will rise

    (c) will fall if non-skilled labor is a normal input and will rise if non-skilled labor is an inferior input.