Economics 540B — Spring 2009

Problem Set 3 Duality Answers

1. The cost function is defined by

$$c(\mathbf{w}, y) = \min_{\mathbf{x}} \{\mathbf{wx} : \mathbf{x} \in V(y)\}$$

where $V(y)$ is the input requirement set.

(a) When is it possible to exactly construct $V(y)$ given the cost function, $c(\mathbf{w}, y)$? How is it done?

It is possible if $V(y)$ is regular, monotonic, and convex. In this case

$$V(y) = V^*(y) = \{\mathbf{x} \geq \mathbf{0} : \mathbf{wx} \geq c(\mathbf{w}, y) \text{ for all } \mathbf{w} \geq \mathbf{0}\}$$

(b) Regardless of whether one can exactly construct $V(y)$ from the cost function, show that we can always construct a set, $V^*(y)$, such that $V(y) \subseteq V^*(y)$.

Again, let

$$V^*(y) = \{\mathbf{x} \geq \mathbf{0} : \mathbf{wx} \geq c(\mathbf{w}, y) \text{ for all } \mathbf{w} \geq \mathbf{0}\}$$

In class we proved:

LEMMA: $V(y) \subseteq V^*(y)$.

Proof: If $\mathbf{x} \in V(y)$ then surely $\mathbf{wx} \geq c(\mathbf{w}, y)$ for all $\mathbf{w} \geq \mathbf{0}$. Hence, $\mathbf{x} \in V^*(y)$.

(c) Here is a quote from your textbook. "*the cost function of a firm summarizes all of the economically relevant aspects of its technology.*" Varian (1992, page 84) Explain what Varian means by this statement. Part of your answer should include an explanation of the term "*economically relevant aspects of its technology*". Be sure to justify the entire quoted statement.

The economically relevant aspects of the technology consists of those input-output vectors (x, y) for which x is cost-minimizing over $V(y)$ for some input price vector, $w \geq 0$. Now recall the following result proved in the text and in class.

If

$$c(\mathbf{w}, y) = \min_{\mathbf{x}} \{\mathbf{wx} : \mathbf{x} \in V(y)\}$$

and, we know, $V(y) \subseteq V^*(y) = \{\mathbf{x} : \mathbf{wx} \geq c(\mathbf{w}, y) \text{ for all } \mathbf{w} \geq \mathbf{0}\}$. Define

$$c^*(\mathbf{w}, y) = \min_{\mathbf{x}} \{\mathbf{wx} : \mathbf{x} \in V^*(y)\}.$$

Theorem: $c^*(\mathbf{w}, y) = c(\mathbf{w}, y)$. Hence, every such input vector, x, that is cost-minimizing over $V(y)$ for some $w \geq 0$, is also cost-minimizing over $V^*(y)$ for the same $w \geq 0$. Since the cost function determines the "outer approximation" $V^*(y)$ it summarizes all of these economically relevant aspects.

There are (x, y) vectors for which x is *not* cost-minimzing over $V(y)$ for any $w \geq 0$. These are *economically irrelevant* input-output vectors. For these vectors, the input vector will lie in either a nonconvex or a nonmonotonic region of $V(y)$. These input vectors will also never be cost-minimizing over $V^*(y)$ for any $w \geq 0$.

The above can be further elaborated if we assume differentiablity of the cost function and thereby use Shephard's lemma. Given $c(w, y)$, then each economically relevant input-output vector is given by

$$(x, y) = \left(\frac{\partial c(w, y)}{\partial w}, y \right),$$

where

$$\frac{\partial c(w, y)}{\partial w} = \left(\frac{\partial c(w, y)}{\partial w_1}, \frac{\partial c(w, y)}{\partial w_2}, \ldots, \frac{\partial c(w, y)}{\partial w_n} \right),$$

and y is in the domain of $c(w, y)$.

2. A firm's cost function is given by

$$c(\mathbf{w}, y) = b_1 w_1 + b_2 w_2 + 2 w_1^{1/2} w_2^{1/2} y, \qquad b_1 > 0, b_2 > 0.$$

   (a) Show that this cost function is monotonic and concave in input prices.

   Compute first derivatives to show monotoniciy. Compute Hessian matrix to show that it is negative semidefinite.

   (b) Find the firm's production function.

   Using Shephard's Lemma

$$\begin{aligned} x_1^* &= \frac{\partial c(\mathbf{w}, y)}{\partial w_1} = b_1 + w_1^{-1/2} w_2^{1/2} y \\ x_2^* &= \frac{\partial c(\mathbf{w}, y)}{\partial w_2} = b_2 + w_1^{1/2} w_2^{-1/2} y \end{aligned}$$

   or

$$\begin{aligned} (x_1 - b_1) &= w_1^{-1/2} w_2^{1/2} y \\ (x_2 - b_2) &= w_1^{1/2} w_2^{-1/2} y \end{aligned}$$

Hence,

$$\begin{aligned}
(x_1 - b_1)(x_2 - b_2) &= y^2 \\
y &= f(x_1, x_2) \\
&= (x_1 - b_1)^{1/2} (x_2 - b_2)^{1/2}
\end{aligned}$$

3. Find the input requirement set, $V(y)$, for each of the following cost functions. You should show all of your work and justify each step in your reasoning.

   (a) $c(\mathbf{w}, y) = \min\left\{\frac{w_1}{a_1}, \frac{w_2}{a_2}\right\} y$

$$\begin{aligned}
V(y) &= \left\{(x_1, x_2) : w_1 x_1 + w_2 x_2 \geq \min\left\{\frac{w_1}{a_1}, \frac{w_2}{a_2}\right\} y \text{ for all } w_1, w_2 > 0\right\} \\
&= \left\{\begin{array}{c} (x_1, x_2) : w_1 x_1 + w_2 x_2 \geq \frac{w_1}{a_1} y \\ \text{and } w_1 x_1 + w_2 x_2 \geq \frac{w_2}{a_2} y \\ \text{for all } w_1, w_2 > 0 \end{array}\right\} \\
&= \left\{\begin{array}{c} (x_1, x_2) : y \leq a_1 x_1 + \frac{a_1 w_2}{w_1} x_2 \\ \text{and } y \leq \frac{a_2 w_1}{w_2} x_1 + a_2 x_2 \\ \text{for all } w_1, w_2 > 0 \end{array}\right\}
\end{aligned}$$

Now

$$\begin{aligned}
\frac{w_2}{a_2} &\leq \frac{w_1}{a_1} \Rightarrow \frac{a_1 w_2}{w_1} \leq a_2 \\
&\Rightarrow y \leq a_1 x_1 + \frac{a_1 w_2}{w_1} x_2 \\
&\leq a_1 x_1 + a_2 x_2
\end{aligned}$$

$$\begin{aligned}
\frac{w_1}{a_1} &\leq \frac{w_2}{a_2} \Rightarrow \frac{a_2 w_1}{w_2} \leq a_1 \\
\Rightarrow y &\leq \frac{a_2 w_1}{w_2} x_1 + a_2 x_2 \\
&\leq a_1 x_1 + a_2 x_2
\end{aligned}$$

Thus

$$V(y) = \{(x_1, x_2) : y \leq a_1 x_1 + a_2 x_2\}$$

(b) $c(\mathbf{w}, y) = (b_1 w_1 + b_2 w_2)\, y$

$$
\begin{aligned}
V(y) &= \{(x_1, x_2) : (b_1 w_1 + b_2 w_2)\, y \leq w_1 x_1 + w_2 x_2 \text{ for all } w_1, w_2 \geq 0\} \\
&= \left\{ \begin{array}{c} (x_1, x_2) : y \leq \dfrac{x_1}{b_1} \text{ for all } w_1 > 0, w_2 \to 0 \\ \\ \text{and } y \leq \dfrac{x_2}{b_2} \text{ for all } w_1 \to 0, w_2 > 0 \end{array} \right\} \\
&= \left\{ (x_1, x_2) : y \leq \frac{x_1}{b_1} \text{ and } y \leq \frac{x_2}{b_2} \right\} \\
&= \left\{ (x_1, x_2) : y \leq \min \left\{ \frac{x_1}{b_1}, \frac{x_2}{b_2} \right\} \right\}.
\end{aligned}
$$

(c) $c(\mathbf{w}, y) = \gamma_1 w_1 + 2 w_1^{1/2} w_2^{1/2} y + \gamma_2 w_2$ This is just like problem 2.

4. According the article by Primont and Sawyer (1993),

$$
c(\mathbf{w}, y) = \min_{\mathbf{x}} \{\mathbf{wx} : f(\mathbf{x}) \geq y\} \tag{1}
$$

if and only if

$$
f(\mathbf{x}) = \min_{\mathbf{w}, y} \{y : \mathbf{wx} \leq c(\mathbf{w}, y)\}. \tag{2}
$$

The corresponding Lagrange functions are

$$
L = \mathbf{wx} + \lambda \left(y - f(\mathbf{x})\right) \tag{3}
$$

and

$$
L^* = y + \kappa \left(\mathbf{wx} - c(\mathbf{w}, y)\right).
$$

(a) Apply the envelope theorem to find

$$
\frac{\partial c(\mathbf{w}, y)}{\partial w_i}, i = 1, ..., n \text{ and } \frac{\partial c(\mathbf{w}, y)}{\partial y}
$$

from (1) and to find

$$
\frac{\partial f(x)}{\partial x_i}, i = 1, ..., n,
$$

from (2).

We get

$$
\frac{\partial c(\mathbf{w}, y)}{\partial w_i} = x_i^*, i = 1, ..., n \tag{4}
$$

$$\frac{\partial c(\mathbf{w}, y)}{\partial y} = \lambda^* \tag{5}$$

$$\frac{\partial f(x)}{\partial x_i} = \kappa w_i^* \tag{6}$$

(b) Give *economic* interpretations for the results in (a).

(4) is just Shephard's Lemma and (5) is the result that the Lagrangian multiplier is marginal cost. (6) is a shadow pricing result. It implies that the relative shadow prices for input vector x are given by

$$\frac{\frac{\partial f(x)}{\partial x_i}}{\frac{\partial f(x)}{\partial x_j}} = \frac{w_i^*}{w_j^*}, i, j = 1, ..., n.$$