Economics 540B Spring 2009
Problem Set 5 Preferences, Utility, Utility Maximization, Marshallian demand.
Short Answers

1. The strict preference ordering, $\succ$, is

   (a) asymmetric: If $x \succ y$ then, by definition, (1) $x \succeq y$ and (2) $y \not\succeq x$. If $y \succ x$ then (3) $y \succeq x$ and (4) $x \not\succeq y$. Clearly, (3) contradicts (2) and (4) contradicts (1). Hence, $x \succ y \Rightarrow y \not\succ x$.

   (b) transitive: If $x \succ y$ and $y \succ z$ then (1) $x \succeq y$ (2) $y \not\succeq x$ (3) $y \succeq z$ and (4) $z \not\succeq y$. Transitivity of $\succeq$, (1), and (3) imply that (5) $x \succeq z$. Suppose $z \succeq x$. This implies, along with transitivity of $\succeq$ and (1) that $z \succeq y$. But this contradicts (4). Conclude that (6) $z \not\succeq x$. Now, (5) and (6) imply that $x \succ z$.

   (c) negatively transitive: Write the transitivity of $\succeq$ as $z \succeq y$ and $y \succeq x \Rightarrow z \succeq x$. The contrapositive statement is

   $$z \not\succeq x \Rightarrow z \not\succeq y \text{ or } y \not\succeq x.$$

   By completeness,

   $$\begin{aligned} z &\not\succeq y \Rightarrow y \succ z \\ y &\not\succeq x \Rightarrow x \succ y \end{aligned}$$

   Hence,

   $$z \not\succeq x \Rightarrow y \succ z \text{ or } x \succ y \tag{1}$$

   Finally,

   $$x \succ z \Rightarrow z \not\succeq x \tag{2}$$

   by definition of $\succ$. Then (1) and (2) imply that

   $$x \succ z \Rightarrow x \succ y \text{ or } y \succ z$$

2. Since $u(x)$ represents $\succeq$,

   $$x \succeq y \Leftrightarrow u(x) \geq u(y) \text{ for all } x, y \text{ in } X.$$

   Complete: We must have $u(x) \geq u(y)$ or $u(y) \geq u(x)$ for all x, y in X.Hence $x \succeq y$ or $y \succeq x$ for all x, y in X.

   Reflexive: Since $u(x) = u(x)$ for all x in X we must have $x \succeq x$ for all x in X.

   Transitive: Since $u(x) \geq u(y)$ and $u(y) \geq u(z)$ implies that $u(x) \geq u(z)$ for all x, y, z in X we must have

   $$x \succeq y \text{ and } y \succeq z \text{ implies that } x \succeq z \text{ for all } x, y, z \text{ in } X.$$

3. Let $\mathcal{U} = \{u : u(x) = u \text{ for some } x \text{ in } X\}$ be the range of the function $u(x)$. Define a mapping, $f : \mathcal{U} \to R$, by

$$f(u) = w(x) \text{ if } u = u(x) \text{ for some } x \text{ in } X.$$

We want to show that i) for every $u \in \mathcal{U}$, $f(u)$ is defined and is unique and ii) f is monotonic on $\mathcal{U}$.

i) By the definition of $\mathcal{U}$, for every $u \in \mathcal{U}$, there exists at least one x in X such that $u = u(x)$ and, hence, $f(u)$ has at least one value for every $u \in \mathcal{U}$. To show that $f(u)$ is unique, suppose not. I.e., suppose $u = u(x) = u(x')$ and $w(x) \neq w(x')$ for $x, x' \in X, x \neq x'$. According to $u, x \sim x'$, and according to $w, x \nsim x'$. This is a contradiction. Hence, $f(u)$ is unique.

ii) Suppose $u \in U, u' \in U, u > u'$. Then, for some $x, x' \in X, u = (x) > u(x') = u', x \succ x'$, and $w(x) > w(x')$ since u and w represent the same preference ordering. We conclude that $f(u) > f(u')$, thus f is monotonic.

4. Homothetic preferences

   (a) If the utility function is homogeneous of degree one then for all x and y in X, $x \sim y \Leftrightarrow u(x) = u(y) \Leftrightarrow ku(x) = ku(y) \Leftrightarrow u(kx) = u(ky) \Leftrightarrow kx \sim ky$. Conversely, suppose $\succeq$ is homothetic.. Using Varian's construction, let $e = (1, 1, ..., 1) \in R^k_+$. Then $u(x)e \sim x$ and $u(kx)e \sim kx$ for all $k \geq 0$, By homotheticity, $u(x)e \sim x \Rightarrow ku(x)e \sim kx$. Since $\sim$ is transitive, $u(kx)e \sim ku(x)e$. Since $\succeq$ is strongly monotonic, $u(kx) = ku(x)$ for all $k \geq 0$.

   (b) All monotonic transformations of $u(x)$ 1) represent the homothetic preference ordering and 2) are homothetic functions since $u(x)$ is homogeneous of degree one.

5. Preference ordering

   (a) Yes, u and w represent the preference ordering since $w = \ln u$ is monotonic for $u \geq 0$.

   (b) Yes, u and w represent the preference ordering since $w = u^3$ is monotonic for $u \geq 0$.

   (c) No, u and w do not represent the preference ordering. According to u, $u(4,4) = 16 > 9 = u(1,9)$, i.e. $(4,4) \succ_u (1,9)$. But according to w, $w(4,4) = 8 < 10 = w(1,9)$, i.e., $(1,9) \succ_w (4,4)$. Hence, u and w do not represent the same preferences.

6. Suppose Tom's Marshallian demand function is $\mathbf{x^T}(\mathbf{p}, m)$ and Jerry's Marshallian demand function is $\mathbf{x^J}(\mathbf{p}, m)$.Then

$$\mathbf{p} \cdot \mathbf{x^T}(\mathbf{p}, m) = m$$

and

$$\mathbf{p} \cdot \mathbf{x^J}(\mathbf{p}, m) = m$$

Moreover,

$$u(\mathbf{x^T}(\mathbf{p}, m)) \geq u(\mathbf{x}) \text{ for all } \mathbf{x} \text{ such that } \mathbf{p} \cdot \mathbf{x} \leq m$$

if and only if

$$f\left(u(\mathbf{x^T}(\mathbf{p}, m))\right) \geq f\left(u(\mathbf{x})\right) \text{ for all } \mathbf{x} \text{ such that } \mathbf{p} \cdot \mathbf{x} \leq m$$

if and only if

$$w(\mathbf{x^T}(\mathbf{p}, m)) \geq w(\mathbf{x}) \text{ for all } \mathbf{x} \text{ such that } \mathbf{p} \cdot \mathbf{x} \leq m.$$

Hence, Tom's Marshallian demand function solves Jerry's utility maximization problem. In similar fashion one can show that Jerry's Marshallian demand function solves Tom's utility maximization problem. We conclude that $\mathbf{x^T}(\mathbf{p}, m) = \mathbf{x^J}(\mathbf{p}, m)$.

7. Find the Marshallian demand functions for the utility function given by

$$\tilde{u}(x_1, x_2) = Ax_1^{b_1} x_2^{b_2}, \qquad b_1 > 0, b_2 > 0, \tag{3}$$

and find the Marshallian demand functions for the utility function given by

$$u(x_1, x_2) = x_1^a x_2^{1-a} \qquad a > 0. \tag{4}$$

They are, respectively,

$$x_1 = \frac{b_1}{b_1 + b_2} \frac{m}{p_1} \text{ and } x_2 = \frac{b_2}{b_1 + b_2} \frac{m}{p_2} \tag{5}$$

and

$$x_1 = a\frac{m}{p_1}, x_2 = (1-a)\frac{m}{p_2}. \tag{6}$$

(a) Obvious.

(b) It is given by $\tilde{u} = Au^{b_1 + b_2}$.

8. The Marshallian demand functions are:

$$x_i = \frac{a_i}{p_i} \frac{m}{\sum a_j} = \frac{a_i m}{p_i \sum a_j}, i = 1, ..., k.$$

9. The Marshallian demand functions are:

$$x_1 = \frac{a\left(m - \sum_{j=1}^{2} p_j \gamma_j\right)}{p_1} + \gamma_1, \quad x_2 = \frac{(1-a)\left(m - \sum_{j=1}^{2} p_j \gamma_j\right)}{p_2} + \gamma_2$$

10. The Marshallian demand functions are:

$$x_i = \frac{\left(\frac{p_i}{a_i}\right)^{\frac{1}{\rho-1}}}{\sum_{j=1}^{n} p_j \left(\frac{p_j}{a_j}\right)^{\frac{1}{\rho-1}}} \left[m - \sum_{j=1}^{n} p_j \gamma_j\right] + \gamma_i, i = 1, ..., n.$$

11. For the form of the indirect utility function note that

$$\begin{aligned} v(p,m) &= \max_x \{u(x) : px = m\} \\ &= \max_x \left\{m \cdot u\left(\frac{x}{m}\right) : p\frac{x}{m} = 1\right\} \\ &= m \max_{x/m} \left\{u\left(\frac{x}{m}\right) : p\frac{x}{m} = 1\right\} \\ &= mv(p,1). \end{aligned} \tag{7}$$

Apply Roy's Identity to (7) to show that $x_i(p,m) = x_i(p,1)m$. This assumes that we have demand functions. If consumer choice is not unique we can show something more general about consumer demand. For each (p,m), let $X(p,m)$ be the *set* of preference maximizers, i.e.,

$$X(p,m) = \{x^* : px^* \leq m \text{ and } x^* \succeq x \text{ for all } x \text{ such that } px \leq m\}$$

Then,

$$\begin{aligned} X(p,tm) &= \{x' : px' \leq tm \text{ and } x' \succeq x \text{ for all } x \text{ such that } px \leq tm\} \\ &= t\left\{\frac{x'}{t} : p\frac{x'}{t} \leq m \text{ and } \frac{x'}{t} \succeq \frac{x}{t} \text{ for all } x \text{ such that } p\frac{x}{t} \leq m\right\} \\ &= t\{\hat{x} : p\hat{x} \leq m \text{ and } \hat{x} \succeq z \text{ for all } x \text{ such that } pz \leq m\} \\ &= tX(p,m) \end{aligned}$$

12. The indirect utility functions and the expenditure functions are:

(a)

$$v(p_1, p_2, m) = \frac{m}{a_1 p_1 + a_2 p_2}.$$

and

$$e(p,u) = (a_1 p_1 + a_2 p_2) u.$$

(b)

$$v\left(p_1, p_2, m\right) = \frac{m}{\min\left\{\dfrac{p_1}{b_1}, \dfrac{p_2}{b_2}\right\}}$$

and

$$e(p, u) = \min\left\{\frac{p_1}{b_1}, \frac{p_2}{b_2}\right\} u.$$

13. Yes. The function $v(p, m)$ is homogeneous of degree zero in (p, m). Moreover,

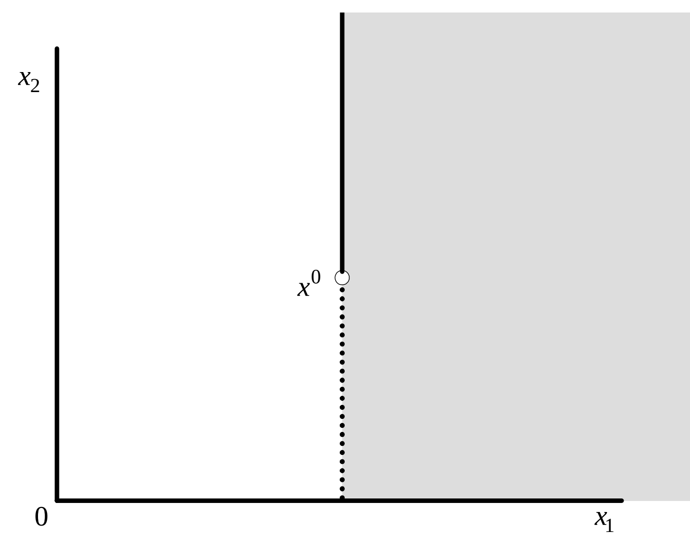
$$\begin{aligned} \frac{\partial v(p,m)}{\partial p_i} &= \frac{p_i}{m}\left(-\frac{m}{p_i^2}\right) = -\frac{1}{p_i} < 0, \quad i = 1, ..., k. \\ \frac{\partial^2 v(p,m)}{\partial p_i^2} &= \frac{1}{p_i^2}, \quad i = 1, ..., k. \\ \frac{\partial v^2(p,m)}{\partial p_i \partial p_j} &= 0, i \neq j \end{aligned}$$

and therefore, $v(p, m)$ is decreasing and convex in prices.

14. Lexicographical preferences.

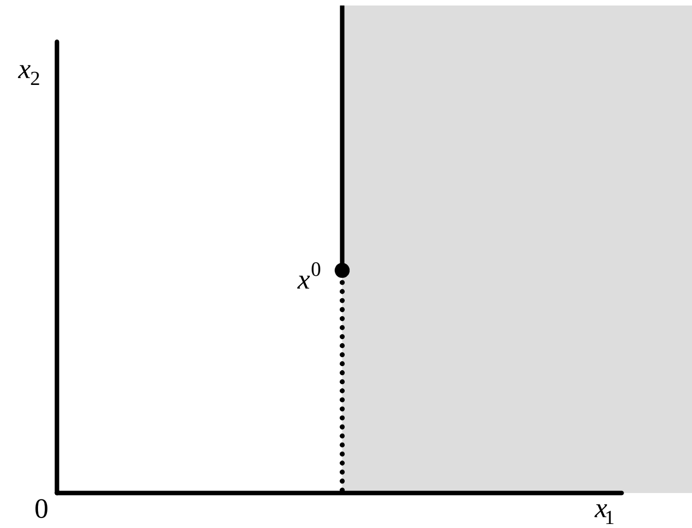
(a) For any commodity bundle, $x^0 = (x_1^0, x_2^0) > 0_2$ sketch the following sets.

i. $\succ (x^0) = \{x = (x_1, x_2) \geq 0_2 : x \succ x^0\}$



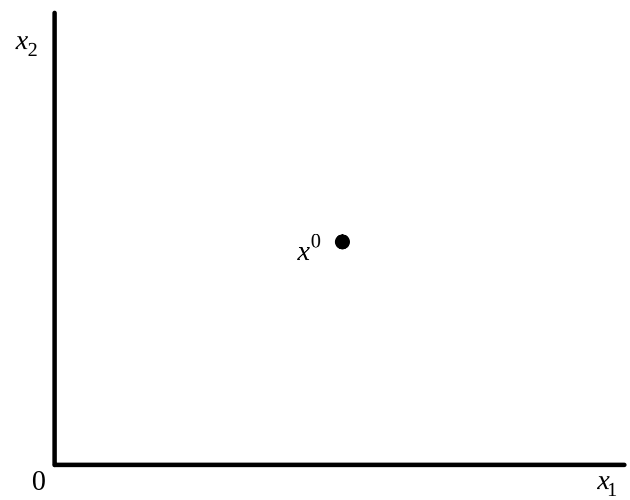
$\succ (x^0)$ includes every bundle above x^0 on the solid line and every bundle to the east of x^0 (the shaded area). It does not include x^0 nor the bundles below x^0 on the dotted line.

ii. $\succeq (x^0) = \{x = (x_1, x_2) \geq 0_2 : x \succeq x^0\}$



$\succeq (x^0) = \succ (x^0) \cup \{x^0\}$

iii. $\sim (x^0) = \{x = (x_1, x_2) \geq 0_2 : x \sim x^0\}$



$\sim (x^0) = \{x^0\}$

(b) The Marshallian demand functions are $x_1^* = m/p_1$ and $x_2^* = 0$.

15. True or False? Justify your answer.

(a) This is TRUE. Reuse the argument in the answer to 6 to show that they have the same Marshallian demands. Then substituting identical demands into identical utility functions must generate identical indirect utility functions.

(b) This is FALSE. Proof by counterexample.

(c) If two consumers have the same complete, transitive, and continuous preferences then they have the same Marshallian demand functions.

This is TRUE. Under the given assumptions they have identical utility functions. Again, reuse the argument in the answer to 6 to show that they have the same Marshallian demands.