Economics 540B — Spring 2009

Problem Set 5 Preferences, Utility, Utility Maximization, Marshallian demand.

1. A weak preference ordering, $\succeq$, is complete, reflexive, and transitive. Show that the associated strict preference ordering, $\succ$, is

   (a) Asymmetric: If $\mathbf{x} \succ \mathbf{y}$ then it is not the case that $\mathbf{y} \succ \mathbf{x}$.

   (b) Transitive: If $\mathbf{x} \succ \mathbf{y}$ and $\mathbf{y} \succ \mathbf{z}$ then $\mathbf{x} \succ \mathbf{z}$.

   (c) Negatively transitive: If $\mathbf{x} \succ \mathbf{z}$ then for all $\mathbf{y}$ in X, either $\mathbf{x} \succ \mathbf{y}$ or $\mathbf{y} \succ \mathbf{z}$.

   HINT 1: Do not appeal to intuition.

   HINT 2: Show that the properties of $\succ$ follow logically from the properties of $\succeq$.

2. Suppose a preference ordering $\succeq$ is represented by a utility function $u(\mathbf{x})$. Show that $\succeq$ must be complete, reflexive, and transitive.

3. Suppose that a preference ordering $\succeq$ is represented by two utility functions, $u(\mathbf{x})$ and $w(\mathbf{x})$. Show that there exists an increasing (monotonic) function f such that $w(\mathbf{x}) = f(u(\mathbf{x}))$.

   Hint: Let $\mathcal{U} = \{u : u(\mathbf{x}) = u \text{ for some } \mathbf{x} \text{ in } X\}$ be the range of the function $u(\mathbf{x})$. Define a mapping, $f : \mathcal{U} \to R$, by

   $$f(u) = w(\mathbf{x}) \text{ if } u = u(\mathbf{x}) \text{ for some } \mathbf{x} \text{ in } X.$$

   Show that i) for every $u \in \mathcal{U}$, $f(u)$ is well defined and is unique, i.e., f is a function, and ii) f is increasing (monotonic) on $\mathcal{U}$.

4. A preference ordering is said to be *homothetic* if for all $\mathbf{x}$ and $\mathbf{y}$ in X we have

   $$\mathbf{x} \sim \mathbf{y} \Leftrightarrow k\mathbf{x} \sim k\mathbf{y} \text{ for all } k \geq 0.$$

   Assume that the preference ordering is complete, transitive, continuous, and strongly monotonic and therefore is represented by a continuous utility function as Varian proves on page 97.

   (a) Prove that this preference ordering is homothetic if and only if it can be represented by a utility function that is homogeneous of degree one.

   (b) Why is such a preference ordering called "homothetic"?

5. For each of the following pairs of utility functions determine whether or not they represent the same preference ordering. Justify your answers.

   (a) $u(x_1, x_2) = x_1^{a_1} x_2^{a_2}$ and $w(x_1, x_2) = a_1 \ln x_1 + a_2 \ln x_2$

(b) $u(x_1, x_2) = x_1^{1/3} x_2^{2/3}$ and $w(x_1, x_2) = x_1 x_2^2$.

(c) $u(x_1, x_2) = x_1 x_2$ and $w(x_1, x_2) = x_1 + x_2$

6. Tom's utility function is $u(\mathbf{x})$ and Jerry's utility function is $w(\mathbf{x})$. Show that Tom and Jerry have the same Marshallian demand function, $\mathbf{x}(\mathbf{p}, m)$, if there exists an increasing function f such that $w(\mathbf{x}) = f(u(\mathbf{x}))$.

7. Find the Marshallian demand functions for the utility function given by

$$\tilde{u}(x_1, x_2) = A x_1^{b_1} x_2^{b_2}, \qquad b_1 > 0, b_2 > 0, \tag{1}$$

and find the Marshallian demand functions for the utility function given by

$$u(x_1, x_2) = x_1^a x_2^{1-a} \qquad a > 0. \tag{2}$$

(a) Show that the utility functions in (1) and (2) yield the same Marshallian demand functions if we assume that

$$a = \frac{b_1}{b_1 + b_2}, \quad (1 - a) = \frac{b_2}{b_1 + b_2}. \tag{3}$$

(b) Given (3), find the monotonic transformation, f, that satisfies $\tilde{u} = f(u)$.

8. Find the Marshallian demand functions for the utility function given by

$$u(\mathbf{x}) = \prod_{i=1}^{k} x_i^{a_i} = x_1^{a_1} \times x_2^{a_2} \times \cdots \times x_k^{a_k}.$$

9. If the consumption of any good falls below a certain "subsistence level" then the consumer ceases to exist and the utility function is undefined (or takes a value of $-\infty$). Nonnegative utility is acquired when the amounts of goods 1 and 2 are both greater than or equal to the subsistence levels given by $\gamma_1 \geq 0$ and $\gamma_2 \geq 0$, respectively. The Cobb-Douglas subsistence level utility function is given by

$$u(x_1, x_2) = (x_1 - \gamma_1)^a (x_2 - \gamma_2)^{1-a} \qquad a > 0. \tag{4}$$

Find the Marshallian demand functions for (4).

10. Suppose the utility function is given by

$$u(\mathbf{x}) \;=\; \left[\sum_{i=1}^{n} a_i \left(x_i - \gamma_i \right)^{\rho} \right]^{\frac{1}{\rho}}, \tag{5}$$

$$\text{where } -\infty \;<\; \rho < 1, \rho \neq 0, \text{ and } \gamma_i \geq 0, \quad i = 1, ..., n.$$

Find the Marshallian demand functions for (5).

11. Suppose the utility function is homogeneous of degree one. Show that the indirect utility function has the following form: $v(\mathbf{p}, m) = v(\mathbf{p}, 1)m$. Also show that each demand function has the form: $x_i(\mathbf{p}, m) = x_i(\mathbf{p}, 1)m$.

12. Find the indirect utility function and the expenditure function for each of the following utility functions.

    (a) $u(x_1, x_2) = \min\{x_1/a_1, x_2/a_2\}, \qquad a_1 > 0, a_2 > 0.$

    (b) $u(x_1, x_2) = b_1 x_1 + b_2 x_2, \qquad b_1 > 0, b_2 > 0.$

13. One of your friends tells you that his indirect utility function is given by $v(\mathbf{p}, m) = \sum_{i=1}^{k} \ln(m/p_i)$. Could this be an indirect utility function, i.e., derived from utility maximization? Justify your answer.

14. Lexicographical preferences in the two good case are defined by a strict preference ordering that is given by:

$$(x_1, x_2) \succ (y_1, y_2) \qquad \text{if} \qquad \begin{cases} x_1 > y_1 \\ \text{or} \\ x_1 = y_1 \quad \text{and} \quad x_2 > y_2 \end{cases}$$

    For this example both commodities are goods.

    (a) For any commodity bundle, $x^0 = (x_1^0, x_2^0) > 0_2$ sketch the following sets.

        i. $\succ (x^0) = \{x = (x_1, x_2) \geq 0_2 : x \succ x^0\}$

        ii. $\succeq (x^0) = \{x = (x_1, x_2) \geq 0_2 : x \succeq x^0\}$

        iii. $\sim (x^0) = \{x = (x_1, x_2) \geq 0_2 : x \sim x^0\}$

    (b) Find the Marshallian demand functions.

15. True or False? Justify your answer.

    (a) If two consumers have the same utility function then they have the same indirect utility function.

    (b) If two consumers have the same indirect utility function then they have the same utility function.

    (c) If two consumers have the same complete, transitive, and continuous preferences then they have the same Marshallian demand functions.