

1. A consumer's initial endowment vector is $\omega = (\omega_1, \dots, \omega_k)$. Her utility functions is Cobb-Douglas:

$$u = \prod_{i=1}^k x_i^{\alpha_i}, \quad \sum_{i=1}^k \alpha_i = 1.$$

As usual, prices are $\mathbf{p} = (p_1, \dots, p_k)$.

Before we answer the questions below let us derive the usual demands when income is just a number m . In this case

$$\begin{aligned} x_i(p, m) &= \frac{\alpha_i}{p_i} m \\ \frac{\partial x_i(p, m)}{\partial p_j} &= \begin{cases} 0 & \text{if } i \neq j \\ -\frac{\alpha_i}{p_i^2} m & \text{if } i = j \end{cases} \\ \frac{\partial x_i(p, m)}{\partial m} &= \frac{\alpha_i}{p_i}. \end{aligned}$$

$$\begin{aligned} v(p, m) &= \prod_{i=1}^k \left(\frac{\alpha_i}{p_i} \right)^{\alpha_i} m \\ m &= e(p, u) = \prod_{i=1}^k \left(\frac{p_i}{\alpha_i} \right)^{\alpha_i} u \\ &= \prod_{\ell=1}^k \left(\frac{p_\ell}{\alpha_\ell} \right)^{\alpha_\ell} u \\ h_i(p, u) &= \frac{\partial e(p, u)}{\partial p_i} = \frac{\alpha_i}{p_i} \prod_{\ell=1}^k \left(\frac{p_\ell}{\alpha_\ell} \right)^{\alpha_\ell} u \end{aligned}$$

$$\begin{aligned} \frac{\partial h_i(p, u)}{\partial p_j} &= \frac{\alpha_i \alpha_j}{p_i p_j} \prod_{\ell=1}^k \left(\frac{p_\ell}{\alpha_\ell} \right)^{\alpha_\ell} u, i \neq j \\ \frac{\partial h_i(p, u)}{\partial p_i} &= \left[\left(-\frac{\alpha_i}{p_i^2} \right) + \left(\frac{\alpha_i}{p_i} \right)^2 \right] \prod_{\ell=1}^k \left(\frac{p_\ell}{\alpha_\ell} \right)^{\alpha_\ell} u \\ &= \frac{\alpha_i(\alpha_i - 1)}{p_i^2} \prod_{\ell=1}^k \left(\frac{p_\ell}{\alpha_\ell} \right)^{\alpha_\ell} u, i = j \\ \frac{\partial x_i(\mathbf{p}, m)}{\partial p_j} &= \frac{\partial h_i(\mathbf{p}, u)}{\partial p_j} - \frac{\partial x_i(\mathbf{p}, m)}{\partial m} x_j(\mathbf{p}, m) \end{aligned}$$

which, in this case is

$$\begin{aligned} 0 &= \frac{\alpha_i \alpha_j}{p_i p_j} \prod_{\ell=1}^k \left(\frac{p_\ell}{\alpha_\ell} \right)^{\alpha_\ell} u - \frac{\alpha_i}{p_i} \left(\frac{\alpha_j}{p_j} m \right) \\ &= \frac{\alpha_i \alpha_j}{p_i p_j} m - \frac{\alpha_i}{p_i} \left(\frac{\alpha_j}{p_j} m \right) \text{ if } i \neq j \end{aligned}$$

and

$$-\frac{\alpha_i}{p_i^2} m = \left(\frac{\alpha_i(\alpha_i - 1)}{p_i^2} \right) m - \frac{\alpha_i}{p_i} \left(\frac{\alpha_i}{p_i} m \right) \text{ if } i = j$$

(a) Show that the Marshallian demand functions are given by

$$x_i(\mathbf{p}, \mathbf{p}\boldsymbol{\omega}) = \frac{\alpha_i}{p_i} \mathbf{p}\boldsymbol{\omega}, \quad i = 1, \dots, k.$$

(b) Derive the Slutsky equation for this Cobb-Douglas consumer.

$$\begin{aligned} \frac{\partial x_i(\mathbf{p}, \mathbf{p}\boldsymbol{\omega})}{\partial p_j} &= \frac{\partial h_i(\mathbf{p}, u)}{\partial p_j} - \frac{\partial x_i(\mathbf{p}, \mathbf{p}\boldsymbol{\omega})}{\partial m} (x_j(\mathbf{p}, \mathbf{p}\boldsymbol{\omega}) - \omega_j) \\ \frac{\alpha_i}{p_i} \omega_j &= \frac{\alpha_i \alpha_j}{p_i p_j} \mathbf{p}\boldsymbol{\omega} - \frac{\alpha_i}{p_i} \left(\frac{\alpha_j}{p_j} \mathbf{p}\boldsymbol{\omega} - \omega_j \right), \quad i \neq j \\ -\frac{\alpha_i}{p_i^2} \mathbf{p}\boldsymbol{\omega} + \frac{\alpha_i}{p_i} \omega_i &= \left(\frac{\alpha_i(\alpha_i - 1)}{p_i^2} \right) \mathbf{p}\boldsymbol{\omega} - \frac{\alpha_i}{p_i} \left(\frac{\alpha_i}{p_i} \mathbf{p}\boldsymbol{\omega} - \omega_i \right), \quad i = j \end{aligned}$$

2. In the two-good case, the demand system in Question 1 is

$$x_1 = \frac{\alpha_1}{p_1} (p_1 \omega_1 + p_2 \omega_2) \tag{1}$$

$$x_2 = \frac{\alpha_2}{p_2} (p_1 \omega_1 + p_2 \omega_2) \tag{2}$$

Find $\partial x_1 / \partial p_1$. Show that $\partial x_1 / \partial p_1 = 0$ when $\omega_2 = 0$.

$$\frac{\partial x_1}{\partial p_1} = -\frac{\alpha_1 p_2 \omega_2}{p_1^2}.$$

3. Varian, Exercise 9.1, page 157. Suppose preferences are homothetic. Show that

$$\frac{\partial x_i(\mathbf{p}, m)}{\partial p_j} = \frac{\partial x_j(\mathbf{p}, m)}{\partial p_i}.$$

Using the Slutsky equation,

$$\begin{aligned} \frac{\partial x_i(\mathbf{p}, m)}{\partial p_j} &= \frac{\partial h_i(p, u)}{\partial p_j} - x_i(p, 1) x_j(p, 1) m \\ &= \frac{\partial x_j(\mathbf{p}, m)}{\partial p_i} \end{aligned}$$

4. Suppose $p = tp^0$, i.e., prices of the x -goods always change in the same proportion for some scalar, t . Let $P = t$ and $X = p^0x$. Then $p = Pp^0$. Define: $V(P, q, m) = \max_{x,z} \{u(x, z) : Pp^0x + qz = m\}$

Claim: This indirect function is homogeneous of degree zero and is quasiconvex and nonincreasing in prices. Moreover,

$$X(P, q, m) = -\frac{\partial V(P, q, m)/\partial P}{\partial V(P, q, m)/\partial m} = p^0x(p, q, m) \quad (3)$$

Proof: Since

$$v(p, q, m) = \max_{x,z} \{u(x, z) : px + qz = m\}$$

it is apparent that

$$V(P, q, m) = v(Pp^0, q, m).$$

And so V “inherits” all of its properties in (P, q) from the the corresponding properties of v in (p, q) . For example,

$$\begin{aligned} V(tP, tq, tm) &= v(tPp^0, tq, tm) \\ &= v(Pp^0, q, m) \\ &= V(P, q, m) \end{aligned}$$

since $v(p, q, m)$ is homogeneous of degree one in prices and income. Quasiconvexity of v implies that

$$\begin{aligned} &\{(p, q) : v(p, q, m) \leq u\} \\ &= \{(Pp^0, q) : v(Pp^0, q, m) \leq u\} \\ &= \{(P, q) : V(P, q, m) \leq u\} \end{aligned}$$

is a convex set. Hence, if

$$v(Pp^0, q, m) \leq u \text{ and } v(P'p^0, q', m) \leq u$$

$$V(P, q, m) \leq u \text{ and } V(P', q', m) \leq u$$

then

$$v(\lambda Pp^0 + (1 - \lambda)P'p^0, \lambda q + (1 - \lambda)q', m) \leq u$$

$$V(\lambda P + (1 - \lambda)P', \lambda q + (1 - \lambda)q', m) \leq u$$

Again, $v(p, q, m) = v(Pp^0, q, m) = V(P, q, m)$. Thus, using the chain rule,

$$\partial V(P, q, m)/\partial P = \sum_i (\partial v(Pp^0, q, m)/\partial p_i) p_i^0.$$

Also,

$$\partial V(P, q, m)/\partial m = \partial v(Pp^0, q, m)/\partial m.$$

Thus,

$$\begin{aligned} -\frac{\partial V(P, q, m)/\partial P}{\partial V(P, q, m)/\partial m} &= -\frac{\sum (\partial v(Pp^0, q, m)/\partial p_i) p_i^0}{\partial v(Pp^0, q, m)/\partial m} \\ &= \sum x_i(Pp^0, q, m) p_i^0 \\ &= p^0 x(Pp^0, q, m) \\ &= p^0 x(p, q, m) \quad (p = Pp^0) \\ &= X(Pp^0, q, m) \end{aligned}$$

This shows that Roy's identity for individual goods implies Roy's identity for the aggregate good under the Hicksian assumption that $p = Pp^0$.

5. For the following utility function verify whether goods 2 and 3 are separable from good 1.

$$u(x_1, x_2, x_3) = x_1^{1/2} x_2^{1/4} x_3^{1/4} + x_1^{1/3} x_2^{1/3} x_3^{1/3}.$$

There are two ways to do this. a) Let $v(x_2, x_3) = x_2^{1/2} x_3^{1/2}$. Then

$$u(x_1, x_2, x_3) = x_1^{1/2} v(x_2, x_3)^{1/2} + x_1^{1/3} v(x_2, x_3)^{2/3}.$$

Or b)

$$\begin{aligned} \frac{\partial u/\partial x_2}{\partial u/\partial x_3} &= \frac{\frac{1}{4} x_1^{1/2} x_2^{-3/4} x_3^{1/4} + \frac{1}{3} x_1^{1/3} x_2^{-2/3} x_3^{1/3}}{\frac{1}{4} x_1^{1/2} x_2^{1/4} x_3^{-3/4} + \frac{1}{3} x_1^{1/3} x_2^{1/3} x_3^{-2/3}} \\ &= \frac{x_3 x_2 \frac{1}{4} x_1^{1/2} x_2^{-3/4} x_3^{1/4} + \frac{1}{3} x_1^{1/3} x_2^{-2/3} x_3^{1/3}}{x_2 x_3 \frac{1}{4} x_1^{1/2} x_2^{1/4} x_3^{-3/4} + \frac{1}{3} x_1^{1/3} x_2^{1/3} x_3^{-2/3}} \\ &= \frac{x_3}{x_2}. \end{aligned}$$

6. Jill likes to drink tea without cream but she also likes to drink coffee with cream. Her utility function is given by $u(x_1, x_2, z) = \min\{x_1, x_2\} \cdot z$, where x_1 is coffee, x_2 is cream, and z is tea

(a) Show that this utility function can be written in the form given by

$$u(x_1, x_2, z) = U(g(x_1, x_2), z) = U(X, z)$$

where X is an aggregate of goods x_1 and x_2 .

Just let $X = g(x_1, x_2) = \min\{x_1, x_2\}$ and let $U(X, z) = Xz$.

(b) Find the demand functions for these three goods.

$$x_1 = x_2 = \frac{m_x}{p_1 + p_2}, \quad z = \frac{m}{2p_z}, \quad X = \frac{m}{2P}$$

$$PX = m_x = \frac{m}{2}, \quad x_1 = x_2 = \frac{m}{2(p_1 + p_2)}$$

(c) Find the indirect utility function for this consumer.

$$v(p_z, p_1, p_2, m) = \left(\frac{m}{2p_z}\right) \left(\frac{m}{2(p_1 + p_2)}\right)$$

$$= \frac{m^2}{4p_z(p_1 + p_2)}$$

7. There are two consumers and two goods. Let x_{ij} be the amount that consumer i consumes of good j , $i, j = 1, 2$. Their utility functions and incomes are

	Consumer 1	Consumer 2
utility function	$u^1(x_{11}, x_{12}) = x_{11}^{1/3} x_{12}^{2/3}$	$u^2(x_{21}, x_{22}) = x_{21}^{2/3} x_{22}^{1/3}$
income	m_1	m_2

Find the aggregate demand for each of the two goods. Does aggregate demand depend on aggregate income?

$$x_{11} = \frac{m_1}{3p_1}, \quad x_{12} = \frac{2m_1}{3p_2}$$

$$x_{21} = \frac{2m_2}{3p_1}, \quad x_{22} = \frac{m_2}{3p_2}$$

Aggregate demands are

$$x_1 = \frac{m_1 + 2m_2}{3p_1} = \frac{1}{p_1} \left(\frac{1}{3}m_1 + \frac{2}{3}m_2 \right)$$

$$x_2 = \frac{2m_1 + m_2}{3p_2} = \frac{1}{p_2} \left(\frac{2}{3}m_1 + \frac{1}{3}m_2 \right)$$

No, aggregate demands do not depend on $m_1 + m_2$.