Economics 540B Spring 2009
Problem Set 8 Uncertainty Answers

1. A person has a utility function of wealth given by $u(w) = \sqrt{w}$, where w is wealth. Her initial wealth is \$100. She also has a lottery ticket that will be worth \$44 with a probability of 1/2 and will be worth \$0 with a probability of 1/2.

   (a) What is her expected utility?

   $$Eu(w) = \frac{1}{2}(100 + 44)^{1/2} + \frac{1}{2}(100)^{1/2} = 11.$$

   (b) What is the lowest price p at which she would part with the lottery ticket?

   $$(100 + p)^{1/2} = 11 \Rightarrow p = \$21.$$

2. SIU Carbondale

   (a) How much insurance will the homeowner purchase?
   The homeowner will maximize expected utility given by

   $$pu(W - L - \pi q + q) + (1 - p)u(W - \pi q)$$

   Differentiate with respect to q and set equal to zero.

   $$pu'(W - L - \pi q^* + q^*)(1 - \pi) - (1 - p)u'(W - \pi q^*)\pi = 0$$

   Rearranging,
   $$\frac{u'(W - L + (1 - \pi)q^*)}{u'(W - \pi q^*)} = \frac{(1 - p)\pi}{p(1 - \pi)}.$$

   For the insurance company, expected profit is given by:

   $$\begin{aligned} & p(\pi q^* - q^*) + (1 - p)\pi q^* \\ = & -p(1 - \pi)q^* + (1 - p)\pi q^* \\ = & \ 0 \end{aligned}$$

   in a competitive market. Thus $\pi = p$. This insurance is actuarially fair. However, since $\pi = p$, the first order condition becomes:

   $$u'(W - L + (1 - \pi)q^*) = u'(W - \pi q^*)$$

   Given strict risk aversion, $u''(W) < 0$, i.e., the first derivative is strictly decreasing then the above condition implies that $W - L + (1 - \pi)q^* = W - \pi q^*$ from which it follows that $q^* = L$.

(b) Now suppose that homeowners have learned over time that when there is a flood, the federal government steps in and offers grants to rebuild the damaged homes. Suppose further that these grants offset a fraction, t, of the total loss L leaving the homeowner to pay for the remaining loss, $(1-t)L$. In this case, how much insurance will the homeowner purchase?

In the above equations just replace L with $(1-t)L$. The last result yields

$$\begin{aligned} W-(1-t)L+(1-\pi)q^* &= W-\pi q^* \\ &\Rightarrow q^*=(1-t)L. \end{aligned}$$

The homeowners only insures that fraction of the damages that are not picked up by the government.

3. Suppose you currently have a job in city A that pays \$1600 per week. If you move to city B and search for a job there is a probability of 1/2 that you will get a job that pays \$2500 per week and a probability of 1/2 that you will get a job that pays \$900 per week. Your utility depends only on your weekly salary . (I.e., except for the salaries, the jobs and the cities are equally attractive and moving costs are zero).

(a) Which option would you choose, i.e. would you stay in city A at your current job or would you move to city B and search for a job?

Your decision is based on your personal attitude toward risk.

(b) A friend named Sue has a utility function given by $u(x)=\sqrt{x}$ where x is her monthly salary. She faces the same options that you do. Which of the above options would Sue choose?

We calculate expected utility under each option.

i. A: $Eu=(1600)^{1/2}=40$
ii. B: $Eu=\frac{1}{2}(2500)^{1/2}+\frac{1}{2}(900)^{1/2}=40.$
Sue is indifferent between the two options.

(c) Given your answers to 3a and 3b, who is more risk averse, you or Sue?

You are more risk averse than Sue if you prefer A and less risk averse if you prefer B.

(d) Another friend named George has a utility function given by $u(x)=\ln x$ where x is his monthly salary. He also faces the same options that you and Sue do. Who is more risk averse, Sue or George? Which of the above options would George choose?

i. A: $Eu=\ln(1600)=7.3778$
ii. B: $Eu=\frac{1}{2}\ln(2500)+\frac{1}{2}\ln(900)=7.3132$
George chooses to stay in city A. He is more risk averse than Sue.

4. A risky alternative pays \$4 with probability 1/2 and pays \$16 with probability 1/2. Find the certainty equivalent of this risky alternative for the Bernoulli utility function, $u(x) = x^{1/2}$.

$$u(CE) = (CE)^{1/2} = \frac{1}{2}(4)^{1/2} + \frac{1}{2}(16)^{1/2} = 3; \qquad CE = 9.$$

5. The Arrow-Pratt measure of *absolute* risk aversion is

$$r(w) = -\frac{u''(w)}{u'(w)}.$$

What is the form of the expected utility function if $r(w) = r =$ a constant.

Just integrate twice.

$$\int -r\, dw = \int \frac{u''(w)}{u'(w)}\, dw$$

$$\begin{aligned} -rw + C &= \ln u'(w) \Rightarrow Ae^{-rw} = u'(w) \\ u'(w) &> 0 \Rightarrow A > 0 \end{aligned}$$

$$u(w) = -\frac{A}{r}e^{-rw} + B$$

or just $u(w) = \frac{-e^{-rw}}{r}$. Then

$$\begin{aligned} u'(w) &= e^{-rw}, u''(w) = -re^{-rw} \\ -\frac{u''(w)}{u'(w)} &= -\frac{-re^{-rw}}{e^{-rw}} = r \end{aligned}$$

6. The Arrow-Pratt measure of *relative* risk aversion is

$$\rho(w) = -\frac{u''(w)w}{u'(w)}.$$

What is the form of the expected utility function if $\rho(w) = \rho =$ a constant.

Just integrate twice.

Rearranging

$$-\frac{\rho}{w} = \frac{u''(w)}{u'(w)}$$

and integrating

$$
\begin{aligned}
-\int \frac{\rho}{w}\,dw &= \int \frac{u''(w)}{u'(w)}\,dw \\
-\rho \ln w + \ln A &= \ln u'(w) \\
Aw^{-\rho} &= u'(w) \\
u(w) &= \begin{cases} \dfrac{A}{1-\rho} w^{1-\rho} + B & \text{if} \quad \rho \neq 1 \\ \\ A \ln w + B & \text{ir} \quad \rho = 1 \end{cases} .
\end{aligned}
$$

or just

$$
u(w) = \begin{cases} \dfrac{w^{1-\rho}}{1-\rho} & \text{if} \quad \rho \neq 1 \\ \\ \ln w & \text{if} \quad \rho = 1 \end{cases}
$$

We would require that $\rho > 0$ so that u is strictly concave (strict risk aversion.)

$$
\begin{aligned}
u &= \frac{w^{1-\rho}}{1-\rho}, u' = w^{-\rho}, u'' = -\rho w^{-\rho-1} \\
-\frac{u'' w}{u'} &= -\frac{-\rho w^{-\rho-1} w}{w^{-\rho}} = \rho > 0
\end{aligned}
$$

Otherwise

$$
\begin{aligned}
u &= \ln w, u' = w^{-1}, u'' = -w^{-2} \\
\rho &= -\frac{-w^{-2} w}{w^{-1}} = 1.
\end{aligned}
$$

7. A consumer has an expected utility function given by $u(w) = w^{1/2}$. She is given the choice of a lottery:

$$
L = p \circ x \oplus (1-p) \circ y
$$

where $x = \$10,000$ and $y = \$0$ or a sure thing, namely $w = \$6400$. Find the value of the probability, p, that makes her indifferent between the lottery L and the sure thing.

We set

$$
\begin{aligned}
u\,(6400) &= p \cdot u(10000) + (1-p) \cdot u(0) \\
80 &= p \cdot 100 + (1-p) \cdot 0 \\
p &= 0.8
\end{aligned}
$$

8. The best lottery, B, is a sure wealth of \$100 and the worst lottery, W, is a sure wealth of \$0. The expected utility function is $u(w) = w^{1/2}$. Find another expected utility, $v(w)$, that represents the same preferences over lotteries and that has the property that $v(W) = 0$ and $v(B) = 1$.

   We seek an affine transformation of the form

   $$v = a + bu, a > 0$$

   with the property that

   $$a + b(0^{1/2}) = 0$$

   and

   $$a + b(100^{1/2}) = 1.$$

   Hence, $a = 0$ and $b = 1/10$. Therefore,

   $$v(w) = \frac{w^{1/2}}{10}.$$

9. This problem is just a variation of the Allais paradox. The solution technique is given in Varian.