Economics 540B — Spring 2009

Problem Set 9 General Equilibrium Answers

1. In a two-consumer, two-good pure exchange economy the two utility functions and initial endowments are given by

$$\begin{aligned} u_1(x_1^1, x_1^2) &= \left(x_1^1\right)^a \left(x_1^2\right)^{1-a} & \boldsymbol{\omega}_1 = \left(\omega_1^1, \omega_1^2\right) \\ u_2(x_2^1, x_2^2) &= \left(x_2^1\right)^b \left(x_2^2\right)^{1-b} & \boldsymbol{\omega}_2 = \left(\omega_2^1, \omega_2^2\right) \end{aligned}$$

where $0 < a < 1, 0 < b < 1$.The demands are given by

$$\begin{aligned} x_1^1 &= \frac{a\left(p_1\omega_1^1 + p_2\omega_1^2\right)}{p_1}, \quad x_1^2 = \frac{(1-a)\left(p_1\omega_1^1 + p_2\omega_1^2\right)}{p_2}, \\ x_2^1 &= \frac{b\left(p_1\omega_2^1 + p_2\omega_2^2\right)}{p_1}, \quad x_2^2 = \frac{(1-b)\left(p_1\omega_2^1 + p_2\omega_2^2\right)}{p_2}. \end{aligned}$$

Aggregate excess demands are given by

$$\begin{aligned} z_1(\mathbf{p}) &= x_1^1 - \omega_1^1 + x_2^1 - \omega_2^1 \\ &= \frac{a\left(p_1\omega_1^1 + p_2\omega_1^2\right) + b\left(p_1\omega_2^1 + p_2\omega_2^2\right)}{p_1} - \left(\omega_1^1 + \omega_2^1\right) \end{aligned}$$

and

$$\begin{aligned} z_2(\mathbf{p}) &= x_1^2 - \omega_1^2 + x_2^2 - \omega_2^2 \\ &= \frac{(1-a)\left(p_1\omega_1^1 + p_2\omega_1^2\right) + (1-b)\left(p_1\omega_2^1 + p_2\omega_2^2\right)}{p_2} - \left(\omega_1^2 + \omega_2^2\right) \end{aligned}$$

Show that $\mathbf{p} \cdot \mathbf{z}(\mathbf{p}) = p_1 z_1(\mathbf{p}) + p_2 z_2(\mathbf{p}) = 0$ (Walras' Law.)

Proof:

$$\begin{aligned} & p_1 z_1(\mathbf{p}) + p_2 z_2(\mathbf{p}) \\ =\; & a\left(p_1\omega_1^1 + p_2\omega_1^2\right) + b\left(p_1\omega_2^1 + p_2\omega_2^2\right) - p_1\left(\omega_1^1 + \omega_2^1\right) \\ & + (1-a)\left(p_1\omega_1^1 + p_2\omega_1^2\right) + (1-b)\left(p_1\omega_2^1 + p_2\omega_2^2\right) - p_2\left(\omega_1^2 + \omega_2^2\right) \\ =\; & p_1\omega_1^1 + p_2\omega_1^2 + p_1\omega_2^1 + p_2\omega_2^2 - p_1\left(\omega_1^1 + \omega_2^1\right) - p_2\left(\omega_1^2 + \omega_2^2\right) \\ =\; & 0 \end{aligned}$$

For good one

$$\begin{aligned} z_1(\mathbf{p}) &= x_1^1 - \omega_1^1 + x_2^1 - \omega_2^1 \\ &= \frac{a\left(p_1\omega_1^1 + p_2\omega_1^2\right) + b\left(p_1\omega_2^1 + p_2\omega_2^2\right)}{p_1} - \left(\omega_1^1 + \omega_2^1\right) = 0 \end{aligned}$$

in a Walrasian equilibrium. Hence

$$a\left(p_1\omega_1^1 + p_2\omega_1^2\right) + b\left(p_1\omega_2^1 + p_2\omega_2^2\right) - p_1\left(\omega_1^1 + \omega_2^1\right) = 0$$

Divide through by p_2 and let $p = p_1/p_2$ to get

$$a\left(p\omega_1^1 + \omega_1^2\right) + b\left(p\omega_2^1 + \omega_2^2\right) - p\left(\omega_1^1 + \omega_2^1\right) = 0.$$

Combine terms involving p to get

$$\left[a\omega_1^1 + b\omega_2^1 - \left(\omega_1^1 + \omega_2^1\right)\right]p + a\omega_1^2 + b\omega_2^2 = 0$$

Simplifying and rearranging

$$\begin{aligned} \left[(1-a)\,\omega_1^1 + (1-b)\omega_2^1\right]p &= a\omega_1^2 + b\omega_2^2 \\ p &= \frac{a\omega_1^2 + b\omega_2^2}{(1-a)\,\omega_1^1 + (1-b)\omega_2^1} \end{aligned}$$

2. In a two-consumer, two-good pure exchange economy the two utility functions are given by

$$\begin{aligned} u_1(x_1^1, x_1^2) &= \left(x_1^1\right)^a \left(x_1^2\right)^{1-a} \qquad \boldsymbol{\omega}_1 = \left(\omega^1, 0\right) \\ u_2(x_2^1, x_2^2) &= \left(x_2^1\right)^b \left(x_2^2\right)^{1-b} \qquad \boldsymbol{\omega}_2 = \left(0, \omega^2\right) \end{aligned}$$

where $0 < a < 1, 0 < b < 1$.The demands are given by

$$\begin{aligned} x_1^1 &= \frac{ap_1\omega^1}{p_1} = a\omega^1, \quad x_1^2 = \frac{(1-a)\,p_1\omega^1}{p_2}, \\ x_2^1 &= \frac{bp_2\omega^2}{p_1}, \quad x_2^2 = \frac{(1-b)\,p_2\omega^2}{p_2} = (1-b)\,\omega^2. \end{aligned}$$

Find the Walrasian equilibrium.

$$\begin{aligned} x_1^1 + x_2^1 &= a\omega^1 + \frac{bp_2\omega^2}{p_1} = \omega^1 \\ a\omega^1\frac{p_1}{p_2} + b\omega^2 &= \omega^1\frac{p_1}{p_2} \\ \left(\omega^1 - a\omega^1\right)\frac{p_1}{p_2} &= b\omega^2 \\ \frac{p_1}{p_2} &= \frac{b\omega^2}{(1-a)\,\omega^1} \end{aligned}$$

or

$$
\begin{aligned}
x_1^2 + x_2^2 &= \frac{(1-a)\, p_1 \omega^1}{p_2} + (1-b)\, \omega^2 = \omega^2 \\
(1-a)\, \omega^1 \frac{p_1}{p_2} &= \omega^2 - (1-b)\, \omega^2 = b\omega^2 \\
\frac{p_1}{p_2} &= \frac{b\omega^2}{(1-a)\, \omega^1}.
\end{aligned}
$$

To get quantities:

$$
\begin{aligned}
x_1^1 &= a\omega^1, \quad x_1^2 = \frac{(1-a)\, p_1 \omega^1}{p_2} = (1-a)\, \omega^1 \left(\frac{b\omega^2}{(1-a)\, \omega^1} \right) = b\omega^2, \\
x_2^1 &= \frac{b p_2 \omega^2}{p_1} = b\omega^2 \frac{(1-a)\, \omega^1}{b\omega^2} = (1-a)\, \omega^1, \quad x_2^2 = (1-b)\, \omega^2.
\end{aligned}
$$

3. (Varian, Exercise 17.4) There are two consumers A and B with the following utility functions and endowments:

$$
\begin{aligned}
u_A\left(x_A^1, x_A^2\right) &= a \ln x_A^1 + (1-a) \ln x_A^2 \quad \boldsymbol{\omega}_A = (0, 1) \\
u_B\left(x_B^1, x_B^2\right) &= \min\left(x_B^1, x_B^2\right) \quad \boldsymbol{\omega}_B = (1, 0)
\end{aligned}
$$

where $0 < a < 1$. Calculate the market clearing prices and the equilibrium allocation.

ANSWER: The demands are

$$
\begin{aligned}
x_A^1 &= \frac{a p_2}{p_1}, \qquad x_A^2 = \frac{(1-a) p_2}{p_2} = 1 - a. \\
x_B^1 &= \frac{p_1}{p_1 + p_2}, \qquad x_B^2 = \frac{p_1}{p_1 + p_2}
\end{aligned}
$$

Then

$$
\begin{aligned}
\frac{a p_2}{p_1} + \frac{p_1}{p_1 + p_2} &= 1 \\
1 - a + \frac{p_1}{p_1 + p_2} &= 1
\end{aligned}
$$

So

$$
\begin{aligned}
\frac{p_1}{p_1 + p_2} &= \frac{p_1/p_2}{p_1/p_2 + 1} \\
&= \frac{p}{p+1} = a
\end{aligned}
$$

$$\begin{aligned} p &= (p+1)\,a = pa + a \\ p &= \frac{a}{1-a}. \end{aligned}$$

$$\begin{aligned} x_A^1 &= \frac{ap_2}{p_1} = \frac{a}{p} = 1-a \\ x_B^1 &= x_B^2 = \frac{p_1}{p_1+p_2} = \frac{p}{p+1} = a \end{aligned}$$

4. There are two consumers, A and B with the following utility functions and endowments:

$$\begin{aligned} u_A\left(x_A^1, x_A^2\right) &= a \ln x_A^1 + (1-a) \ln x_A^2 \quad \boldsymbol{\omega}_A = (0,1) \\ u_B\left(x_B^1, x_B^2\right) &= x_B^1 + x_B^2 \quad \boldsymbol{\omega}_B = (1,0) \end{aligned}$$

where $0 < a < 1$. Calculate the market clearing prices and the equilibrium allocation.

ANSWER: The demands for consumer A are

$$x_A^1 = a\frac{p_2}{p_1} = \frac{a}{p} \qquad x_A^2 = (1-a)\frac{p_2}{p_2} = 1-a$$

Consumer A will always demand a positive amount of each good if p is finite and positive. This requires that consumer B also demands a positive amount of each good. This can only occur if $p = MRS_B = 1$. Hence $x_A^1 = a,\ x_A^2 = 1-a,\ x_B^1 = 1-a,\ x_B^2 = a.$

5. There are two consumers, A and B with the following utility functions and endowments:

$$\begin{aligned} u_A\left(x_A^1, x_A^2\right) &= \min\left(x_A^1, x_A^2\right) \quad \boldsymbol{\omega}_A = (0,3) \\ u_B\left(x_B^1, x_B^2\right) &= x_B^1 + 2x_B^2 \quad \boldsymbol{\omega}_B = (3,0) \end{aligned}$$

Calculate the market clearing prices and the equilibrium allocation.

ANSWER: Consumer A will always demand equal and positive amounts of each good. This requires that consumer B also demands a positive amount of each good. This can only occur if $p = MRS_B = 1/2$. Consumer A's demands are

$$\begin{aligned} x_A^1 &= x_A^2 = \frac{m}{p_1+p_2} \\ &= \frac{3p_2}{p_1+p_2} \\ &= \frac{3}{p+1} \\ &= \frac{3}{\frac{1}{2}+1} = 2 \end{aligned}$$

Hence,

$$x_B^1 = x_B^2 = 1.$$

6. A pure exchange two-good economy consists of n identical consumers with utility functions given by

$$u(x_i^1, x_i^2) = \left(x_i^1\right)^2 + \left(x_i^2\right)^2, \quad i = 1, ..., n.$$

Each consumer has an initial endowment given by

$$(\omega_i^1, \omega_i^2) = (1, 1).$$

(a) Show that the demand function for each good for each consumer is discontinuous at $p_1/p_2 = 1$. (They also fail to be single-valued functions at $p_1/p_2 = 1$.)

When $p_1/p_2 < 1$, then

$$x_i^1 = \frac{p_1 + p_2}{p_1} \text{ and } x_i^2 = 0 \tag{1}$$

When $p_1/p_2 < 1$, then

$$x_i^1 = 0 \text{ and } x_i^2 = \frac{p_1 + p_2}{p_2} \tag{2}$$

When $p_1/p_2 = 1$, then demand is not single-valued - either (1) or (2) is demanded.

(b) Suppose n is an even number. Find the competitive equilibrium. (Hint: Start with the case $n = 2$. Draw the Edgeworth box and find the competitive equilibrium. Then generalize your result to $n = 2z$ where z is a positive integer.

With two consumers there are two units of each good. This requires that $p_1/p_2 = 1$. One consumer ends up with two units of good 1 and the other consumer ends up with two units of good 2. For $n = 2z$, again $p_1/p_2 = 1$. There are n units of each good. One half of the consumers (z) get two units of good 1 and the other half get two units of good 2.

(c) Suppose n is an odd number. Show that a competitive equilibrium does not exist. (Hint: Begin with $n = 2z$ where z is a positive integer and then add one more consumer so that $n = 2z + 1$. What happens?)

With an odd number, the number of suppliers of good one cannot equal the number of buyers of good one. No equilibrium.

7. (Varian, Exercise 17.2) Draw an Edgeworth box example with an infinite number of (relative) prices that are Walrasian equilibria.

8. You are the chief of a tribe of 100 individuals. Your loyal subjects have harvested the only two goods that are consumed in your society. After giving you a tribute of each good the remaining quantities are 500 coconuts (for food) and 300 yards of fiber (for clothing). As a wise and benevolent chief how will you allocate these two goods among your subjects? Justify your answers. The more answers that you can justify the more interesting will be your response.

9. Pure Exchange Economy:

   (a) For a pure exchange economy, define the following terms: feasible allocation, weakly Pareto efficient, strongly Pareto efficient, contract curve, Walrasian equilibrium, and aggregate excess demand. State Walras' Law. Under what assumption does Walras' Law hold?

   (b) Suppose there are two consumers with utility functions

   $$u_1 = x_{11}^{1/3} x_{12}^{2/3} \text{ and } u_2 = \min\{x_{21}, x_{22}\}$$

   and initial allocations

   $$(\omega_{11}, \omega_{12}) = (12, 0) \text{ and } (\omega_{21}, \omega_{22}) = (0, 24).$$

      i. Find the contract curve.
      Consumer 2 will always want to consume equal amounts of the two goods and hence along the contract curve we must have $x_{21} = x_{22}$. This will the case for

      $$0 \leq x_{21} = x_{22} \leq 12,$$

      since there are only 12 units of good 1 in the economy.

      ii. Find the equilibrium price ratio p_1/p_2.
      The demands for consumer 1 are

      $$\begin{aligned} x_{11} &= \frac{12p_1}{3p_1} = 4 \\ x_{12} &= \frac{2(12p_1)}{3p_2} \end{aligned}$$

      and hence $x_{21} = 12 - 4 = 8 = x_{22}$.Whence

      $$\begin{aligned} x_{12} &= \frac{2(12p_1)}{3p_2} = 8\frac{p_1}{p_2} = 16 \\ \frac{p_1}{p_2} &= 2 \end{aligned}$$

iii. Find the four equilibrium quantities, $x_{11}, x_{12}, x_{21}, x_{22}$.
As we have seen

$$\begin{aligned} x_{11} &= 4, x_{12} = 16 \\ x_{21} &= 8 = x_{22}. \end{aligned}$$

10. EQUILIBRIUM WITH PRODUCTION:

(a) Each consumer in a small closed economy has a utility function

$$u(b, g) = \sqrt{b} + \sqrt{g}$$

where b is butter and g is guns. Each consumer also supplies one unit of labor for which they receive a labor wage equal to w. Set up the utility maximization problem and show that the Marshallian demands are given by

$$b = \frac{p_g}{p_b(p_b + p_g)} w \text{ and } g = \frac{p_b}{p_g(p_b + p_g)} w.$$

ANSWER:

$$\max b^{1/2} + g^{1/2} \text{ subject to } p_b b + p_g g = w$$

$$\begin{aligned} \frac{1}{2} b^{-1/2} &= \lambda p_b \\ \frac{1}{2} g^{-1/2} &= \lambda p_g \end{aligned}$$

$$\begin{aligned} \frac{1}{2 p_b b^{1/2}} &= \frac{1}{2 p_g g^{1/2}} = \lambda \\ b &= \frac{1}{4 p_b^2 \lambda^2}, \; g = \frac{1}{4 p_g^2 \lambda^2} \end{aligned}$$

$$\begin{aligned} p_b b &= \frac{1}{4 p_b \lambda^2} \\ p_g g &= \frac{1}{4 p_g \lambda^2} \end{aligned}$$

$$\begin{aligned} w &= \frac{1}{4 p_b \lambda^2} + \frac{1}{4 p_g \lambda^2} \\ w &= \frac{1}{\lambda^2} \left(\frac{p_b + p_g}{4 p_b p_g} \right) \\ \frac{1}{\lambda^2} &= \left(\frac{4 p_b p_g}{p_b + p_g} \right) w \end{aligned}$$

$$
\begin{aligned}
b &= \frac{1}{4p_b^2\lambda^2} = \frac{1}{4p_b^2}\left(\frac{4p_bp_g}{p_b+p_g}\right)w \\
b &= \frac{p_g}{p_b\left(p_b+p_g\right)}w
\end{aligned}
$$

$$
\begin{aligned}
g &= \frac{1}{4p_g^2\lambda^2} = \frac{1}{4p_g^2}\left(\frac{4p_bp_g}{p_b+p_g}\right)w \\
g &= \frac{p_b}{p_g\left(p_b+p_g\right)}w
\end{aligned}
$$

(b) Each firm in this perfectly competitive economy can produce both butter and guns. The production functions are

$$
b = 4L_b \text{ and } g = 2L_g,
$$

where L_b and L_g are the quantities of labor allocated to butter and guns, respectively. We will assume, without loss of generality, that there is only one competitive firm. Why is this assumption justified?

It is justified by constant returns to scale.

(c) Suppose there are 120 consumers. Find the competitive equilibrium price ratios. Find the total quantity of butter and guns produced. How much labor is allocated to each output?

$$
\begin{aligned}
L_b &= \frac{1}{4}b \text{ and } L_g = \frac{1}{2}g \\
\frac{1}{4}b + \frac{1}{2}g &= 120 \\
b + 2g &= 480
\end{aligned}
$$

The PPF is a straight line with slope -1/2. Thus

$$
\frac{p_b}{p_g} = \frac{1}{2}
$$

Alternatively, we know that profit must be zero given constant returns to scale. Thus

$$
\begin{aligned}
p_b b - wL_b &= 0 \\
p_g g - wL_g &= 0
\end{aligned}
$$

Substituting the production functions:

$$
\begin{aligned}
p_b(4L_b) - wL_b &= 0 \\
p_g\left(2L_g\right) - wL_g &= 0
\end{aligned}
$$

Rearranging

$$\begin{aligned}\frac{w}{p_b} &= 4\\ \frac{w}{p_g} &= 2\end{aligned}$$

and so

$$\frac{p_b}{p_g} = \frac{\frac{w}{p_g}}{\frac{w}{p_b}} = \frac{2}{4} = \frac{1}{2}.$$

The aggregate demand for butter is

$$\begin{aligned}B &= \frac{120p_g}{p_b\left(p_b+p_g\right)}w = \frac{120}{\frac{p_b}{p_g}\left(p_b+p_g\right)}w\\ &= \frac{120}{\frac{p_b}{p_g}\left(\frac{p_b}{p_g}+1\right)}\frac{w}{p_g} = \frac{120}{\frac{1}{2}\left(\frac{1}{2}+1\right)}2\\ &= (4)\left(\frac{2}{3}\right)(120) = 320\end{aligned}$$

and the aggregate demand for guns is

$$\begin{aligned}G &= \frac{120p_b}{p_g\left(p_b+p_g\right)}w = \frac{120\left(\frac{p_b}{p_g}\right)}{p_b+p_g}w\\ &= \frac{120\left(\frac{p_b}{p_g}\right)}{\frac{p_b}{p_g}+1}\frac{w}{p_g} = \frac{120\left(\frac{1}{2}\right)}{\frac{1}{2}+1}(2)\\ &= 80\end{aligned}$$

Since

$$b = 4L_b \text{ and } g = 2L_g,$$

$$320 = 4L_b \text{ and } 80 = 2L_g$$

$$L_b = 80 \text{ and } L_g = 40.$$