Economics 540B

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1. (10) A firm's technology is given by

$$T = \left\{(x_1, x_2, y_1, y_2) \geq 0_4 : b_1 y_1 + b_2 y_2 \leq x_1^{a_1} x_2^{a_2}\right\},$$

where $a_1 + a_2 = a$.

(a) Derive the firm's input distance function. Show all of your work.

$$\begin{aligned} D_i(y,x) &= \sup_{\lambda}\left\{\lambda : b_1 y_1 + b_2 y_2 \leq \left(\frac{x_1}{\lambda}\right)^{a_1}\left(\frac{x_2}{\lambda}\right)^{a_2}\right\} \\ &= \sup_{\lambda}\left\{\lambda : \lambda^{a_1+a_2} \leq \frac{x_1^{a_1} x_2^{a_2}}{b_1 y_1 + b_2 y_2}\right\} \\ &= \sup_{\lambda}\left\{\lambda : \lambda \leq \frac{x_1^{a_1/a} x_2^{a_2/a}}{(b_1 y_1 + b_2 y_2)^{1/a}}\right\} \\ &= \frac{x_1^{a_1/a} x_2^{a_2/a}}{(b_1 y_1 + b_2 y_2)^{1/a}} \end{aligned}$$

(b) Write the formula for technical input efficiency for this example.

$$T_i(y,x) = \frac{1}{D_i(y,x)} = \frac{(b_1 y_1 + b_2 y_2)^{1/a}}{x_1^{a_1/a} x_2^{a_2/a}}.$$

2. (15) A weak preference ordering, $\succeq$, is complete, reflexive, and transitive. Show that the associated indifference relation, $\sim$, is

(a) Symmetric: If $\mathbf{x} \sim \mathbf{y}$ then $\mathbf{y} \sim \mathbf{x}$.

$$\begin{aligned} \mathbf{x} \sim \mathbf{y} &\Leftrightarrow \mathbf{x} \succeq \mathbf{y} \ \& \ \mathbf{y} \succeq \mathbf{x} \\ &\Leftrightarrow \mathbf{y} \succeq \mathbf{x} \ \& \ \mathbf{x} \succeq \mathbf{y} \\ &\Leftrightarrow \mathbf{y} \sim \mathbf{x}. \end{aligned}$$

(b) Transitive: If $\mathbf{x} \sim \mathbf{y}$ and $\mathbf{y} \sim \mathbf{z}$ then $\mathbf{x} \sim \mathbf{z}$.

$$\begin{aligned} \mathbf{x} &\sim \mathbf{y} \Leftrightarrow (\mathbf{1})\ \mathbf{x} \succeq \mathbf{y} \ \& \ (\mathbf{2})\ \mathbf{y} \succeq \mathbf{x} \\ \mathbf{y} &\sim \mathbf{z} \Leftrightarrow (\mathbf{3})\ \mathbf{y} \succeq \mathbf{z} \ \& \ (\mathbf{4})\ \mathbf{z} \succeq \mathbf{y} \end{aligned}$$

Transitivity of $\succeq$, (1) and (3) imply that (5) $\mathbf{x} \succeq \mathbf{z}$. Transitivity of $\succeq$, (4) and (2) imply that (6) $\mathbf{z} \succeq \mathbf{x}$. Then (5) and (6) imply that $\mathbf{x} \sim \mathbf{z}$.

3. (20) The following utility function,

$$u = \min\left\{\frac{x_1 - \gamma_1}{a_1}, \frac{x_2 - \gamma_2}{a_2}\right\}, \quad a_1, a_2 > 0, \quad \gamma_1, \gamma_2 > 0,$$

is defined on the consumption set $X = \{(x_1, x_2) : x_1 \geq \gamma_1, x_2 \geq \gamma_2\}$. The consumer has enough money m to buy some of the bundles in X, at prices, p_1 and p_2, i.e., $m > p_1\gamma_1 + p_2\gamma_2$. Derive the consumer's Marshallian demand functions. Then derive the consumer's indirect utility function and the consumer's expenditure function.

Let

$$z_1 = x_1 - \gamma_1 \text{ and } z_2 = x_2 - \gamma_2$$

Then $pz = m - px = y$.The utility function is now:

$$u = \min\left\{\frac{z_1}{a_1}, \frac{z_2}{a_2}\right\}$$

$$\begin{aligned}
\frac{z_1}{a_1} &= \frac{z_2}{a_2}; z_2 = a_2\frac{z_1}{a_1} \\
p_1 z_1 + p_2\frac{a_2}{a_1}z_1 &= y \\
a_1 p_1 z_1 + a_2 p_2 z_1 &= a_1 y
\end{aligned}$$

$$\begin{aligned}
z_1 &= \frac{a_1 y}{a_1 p_1 + a_2 p_2}; \quad z_2 = \frac{a_2 y}{a_1 p_1 + a_2 p_2} \\
x_1^* &= \frac{a_1(m - p_1\gamma_1 - p_2\gamma_2)}{a_1 p_1 + a_2 p_2} + \gamma_1 \\
x_2^* &= \frac{a_2(m - p_1\gamma_1 - p_2\gamma_2)}{a_1 p_1 + a_2 p_2} + \gamma_2
\end{aligned}$$

The indirect utility function is defined by

$$\begin{aligned}
v(p, m) &= \min\left\{\frac{x_1^* - \gamma_1}{a_1}, \frac{x_2^* - \gamma_2}{a_2}\right\} \\
&= \min\left\{\frac{m - p_1\gamma_1 - p_2\gamma_2}{a_1 p_1 + a_2 p_2}, \frac{m - p_1\gamma_1 - p_2\gamma_2}{a_1 p_1 + a_2 p_2}\right\} \\
&= \frac{m - p_1\gamma_1 - p_2\gamma_2}{a_1 p_1 + a_2 p_2}
\end{aligned}$$

The expenditure function is found by solving

$$u = \frac{e(p, u) - p_1\gamma_1 - p_2\gamma_2}{a_1 p_1 + a_2 p_2}$$

to get

$$e(p, u) = (a_1 p_1 + a_2 p_2)\, u + p_1\gamma_1 + p_2\gamma_2$$

4. (15) You observe a consumer making the following choices at the following prices in two different periods.

	p_1	p_2	x_1	x_2
Period 2	1	2	12	6
Period 1	2	1	8	8

Is this consumer maximizing utility? Justify your answer.

We have $p^1 = (2, 1), p^2 = (1, 2), x^1 = (8, 8)$, and $x^2 = (12, 6)$. So

$$\begin{bmatrix} p^1x^1 & p^1x^2 \\ p^2x^1 & p^2x^2 \end{bmatrix} = \begin{bmatrix} 24 & 30 \\ 24 & 24 \end{bmatrix}$$

Since $p^2x^2 = p^2x^1$ $(x^2R^Dx^1)$ and $p^1x^1 < p^1x^2$ $(\text{not}(x^1P^Dx^2))$ WARP and GARP are satisfied. These observations are *consistent* with utility maximization. They do not prove that the consumer is maximizing utility.

5. (15) It can be shown that a direct utility function of the form $v(x_1.x_2) = x_1^{1/2}x_2^{1/2}$ has an expenditure function of the form $e(p, v) = 2p_1^{1/2}p_2^{1/2}v$. Armed with this fascinating information, find the Marshallian demand functions for the following utility function.

$$u(x_1, x_2, z) = \min\left\{x_1^{1/2}x_2^{1/2}, z\right\}$$

The Marshallian demands for $v(x_1.x_2) = x_1^{1/2}x_2^{1/2}$ are

$$x_1 = \frac{m_x}{2p_1} \text{ and } x_2 = \frac{m_x}{2p_2}$$

where m_x is the money allocated to the x-goods.

Let $P = e(p, 1) = 2p_1^{1/2}p_2^{1/2}$ and $X = x_1^{1/2}x_2^{1/2}$. Solve

$$\max_{X,z} \min\{X, z\} \text{ subject to } PX + qz = m.$$

This Leontief utility function has a solution

$$X = z = \frac{m}{P + q}.$$

Then

$$m_x = PX = \frac{mP}{P + q}$$

The Marshallian demands are

$$
\begin{aligned}
x_1 &= \frac{mP}{2p_1(P+q)} = \frac{2p_1^{1/2}p_2^{1/2}m}{2p_1\left(2p_1^{1/2}p_2^{1/2}+q\right)} \\
&= \frac{p_1^{1/2}p_2^{1/2}m}{p_1\left(2p_1^{1/2}p_2^{1/2}+q\right)} \\
x_2 &= \frac{mP}{2p_2(P+q)} = \frac{2p_1^{1/2}p_2^{1/2}m}{2p_2\left(2p_1^{1/2}p_2^{1/2}+q\right)} \\
&= \frac{p_1^{1/2}p_2^{1/2}m}{p_2\left(2p_1^{1/2}p_2^{1/2}+q\right)} \\
z &= \frac{m}{P+q} = \frac{m}{2p_1^{1/2}p_2^{1/2}+q}
\end{aligned}
$$

Note that we check to verify that our answers "add up" in the budget constraint.

$$
\begin{aligned}
p_1x_1+p_2x_2+qz &= \frac{p_1^{1/2}p_2^{1/2}m}{2p_1^{1/2}p_2^{1/2}+q} + \frac{p_1^{1/2}p_2^{1/2}m}{2p_1^{1/2}p_2^{1/2}+q} + \frac{qm}{2p_1^{1/2}p_2^{1/2}+q} \\
&= \frac{2p_1^{1/2}p_2^{1/2}m}{2p_1^{1/2}p_2^{1/2}+q} + \frac{qm}{2p_1^{1/2}p_2^{1/2}+q} \\
&= \frac{\left(2p_1^{1/2}p_2^{1/2}+q\right)m}{2p_1^{1/2}p_2^{1/2}+q} \\
&= m.
\end{aligned}
$$

6. (10) Al's utility function of wealth is $A(w) = w^{1/2}$ and Betty's utility function of wealth is $B(w) = \ln w$. Which individual is more risk averse? Justify your answer.

$$
\begin{aligned}
r_A(w) &= -\frac{A''(w)}{A'(w)} = -\frac{-\frac{1}{4}w^{-3/2}}{\frac{1}{2}w^{-1/2}} = \frac{1}{2w} \\
r_B(w) &= -\frac{B''(w)}{B'(w)} = -\frac{-w^{-2}}{w^{-1}} = \frac{1}{w}
\end{aligned}
$$

Since $r_B(w) = \dfrac{1}{w} > \dfrac{1}{2w} = r_A(w)$, Betty is more risk averse than Al.

7. (15) Marshallian cross-price derivatives

(a) Suppose preferences are homothetic. Show that

$$\frac{\partial x_i\left(\mathbf{p},m\right)}{\partial p_j}=\frac{\partial x_j\left(\mathbf{p},m\right)}{\partial p_i},\quad i,j=1,...,n.$$

We have shown in the Problem Sets that if preferences are homothetic, Marshalllian demands have the property that $x_i(\mathbf{p},m)=x_i(\mathbf{p},1)m, i=1,...,n$. Thus

$$\frac{\partial x_i(\mathbf{p},m)}{\partial m}=x_i(\mathbf{p},1)$$

From the Slutsky equation

$$\begin{aligned}\frac{\partial x_i\left(\mathbf{p},m\right)}{\partial p_j} &= \frac{\partial h_i(p,u)}{\partial p_j}-\frac{\partial x_i(\mathbf{p},m)}{\partial m}x_j(\mathbf{p},m)\\ &= \frac{\partial h_i(p,u)}{\partial p_j}-x_i(\mathbf{p},1)x_j(\mathbf{p},1)m\\ &= \frac{\partial h_j(p,u)}{\partial p_i}-x_j(\mathbf{p},1)x_i(\mathbf{p},1)m\\ &= \frac{\partial h_j(p,u)}{\partial p_i}-\frac{\partial x_j(\mathbf{p},m)}{\partial m}x_i(\mathbf{p},m)\\ &= \frac{\partial x_j\left(\mathbf{p},m\right)}{\partial p_i},\quad i,j=1,...,n.\end{aligned}$$

using the fact that $\frac{\partial h_i(p,u)}{\partial p_j}=\frac{\partial h_j(p,u)}{\partial p_i}$.

(b) We will now attempt the start of a proof of the converse to part (a). We will proceed in steps.

i. Show that the income elasticities of the Marshallian demands are all equal to each other if

$$\frac{\partial x_i\left(\mathbf{p},m\right)}{\partial p_j}=\frac{\partial x_j\left(\mathbf{p},m\right)}{\partial p_i},\quad i,j=1,...,n.$$

Again, since the substitution terms are symmetric the above equation implies that the income effects are also symmetric, i.e.,

$$\frac{\partial x_i(\mathbf{p},m)}{\partial m}x_j(\mathbf{p},m)=\frac{\partial x_j(\mathbf{p},m)}{\partial m}x_i(\mathbf{p},m),\quad i,j=1,...,n.$$

Rearranging

$$\frac{\partial x_i(\mathbf{p},m)}{\partial m}\frac{1}{x_i(\mathbf{p},m)}=\frac{\partial x_j(\mathbf{p},m)}{\partial m}\frac{1}{x_j(\mathbf{p},m)},\quad i,j=1,...,n.$$

and multiplying both sides by m we get

$$\frac{\partial x_i(\mathbf{p},m)}{\partial m}\frac{m}{x_i(\mathbf{p},m)} = \frac{\partial x_j(\mathbf{p},m)}{\partial m}\frac{m}{x_j(\mathbf{p},m)}, \quad i,j = 1,...,n.$$

Hence, all of the income elasticities are equal.

ii. Show that the income elasticities of the Marshallian demands are all equal to unity if they are all equal to each other.
Differentiate the budget identity

$$\sum_{i=1}^{n} p_i x_i(\mathbf{p},m) \equiv m$$

with respect to m. We get

$$\sum_{i=1}^{n} p_i \frac{\partial x_i(\mathbf{p},m)}{\partial m} \equiv 1$$

This implies that

$$\sum_{i=1}^{n} \frac{p_i x_i(\mathbf{p},m)}{m}\frac{\partial x_i(\mathbf{p},m)}{\partial m}\frac{m}{x_i(\mathbf{p},m)} \equiv 1.$$

Since all of the income elasticities are equal this implies that

$$\frac{\partial x_i(\mathbf{p},m)}{\partial m}\frac{m}{x_i(\mathbf{p},m)}\sum_{i=1}^{n}\frac{p_i x_i(\mathbf{p},m)}{m} \equiv 1.$$

So

$$\frac{\partial x_i(\mathbf{p},m)}{\partial m}\frac{m}{x_i(\mathbf{p},m)} \equiv 1, \quad i = 1,...,n.$$

since

$$\sum_{i=1}^{n}\frac{p_i x_i(\mathbf{p},m)}{m} \equiv 1.$$

iii. Show that the Marshallian demands have the form $x_i(\mathbf{p},m) = x_i(\mathbf{p},1)m$ if all of their income elasticities are equal to unity.
The unitary income elasticities may be written as

$$\frac{\partial \ln x_i(\mathbf{p},m)}{\partial \ln m} = 1, \quad i = 1,...,n.$$

Integrating

$$\begin{aligned}\int \frac{\partial \ln x_i(\mathbf{p},m)}{\partial \ln m}\, d\ln m &= \int d\ln m \\ \ln x_i(\mathbf{p},m) &= \ln m + \ln c_i(\mathbf{p})\end{aligned}$$

where $\ln c_i(\mathbf{p})$ is the constant of integration and it depends on prices (but not income). Exponentiating

$$x_i(\mathbf{p}, m) = c_i(\mathbf{p})m.$$

Thus

$$x_i(\mathbf{p}, 1) = c_i(\mathbf{p})$$

and

$$x_i(\mathbf{p}, m) = x_i(\mathbf{p}, 1)m, \quad i = 1, ..., n.$$