Derivative Properties of Directional Technology Distance Functions

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1 Introduction

Properties of the directional technology distance function have been given in a paper by Chambers, Chung, and Färe (1998). This function, $\vec{D}(x, y; g_x, g_y)$, is an implicit representation of an M-output, N-input production technology. An input-output vector, (x, y), is feasible if and only if $\vec{D}(x, y; g_x, g_y) \geqq 0$, where (g_x, g_y) is a "direction" vector to be described later. An important antecedent of the directional technology distance function is the shortage function, introduced by Luenberger (1992, 1995).

In this paper the theory of the directional technology distance function is extended by deriving a set of restrictions on the first and second derivatives of the directional technology distance functions. These restrictions would be useful in building an econometric model based on the directional technology distance function. It is then shown that the usual comparative static results for a competitive firm are easily established. In the final section we present flexible functional forms for estimating directional technology distance functions and some of the required parametric restrictions.

Let $x \in R_+^N$ be the input vector and let $y \in R_+^M$ be the output vector. The technology T is given by

$$T = \{(x, y) : \ x \text{ can produce } y\} .$$

Assume (see Chambers, Chung, Färe (1998))

T1. T is closed
T2. Free disposability: if $(x, y) \in T, x' \geq x$, and $y' \leq y$ then $(x', y') \in T$.
T3. No free lunch: if $(x, y) \in T$ and $x = 0$ then $y = 0$.
T4. Possibility of inaction: $(0, 0) \in T$.
T5. T is convex.

The directional technology distance function is a particular representation of a multi-output, multi-input production technology. Following Chambers, Chung, and Färe (1998),

$$\vec{D}(x,y;g_x,g_y) = \begin{cases} \max\{\beta : (x,y) + \beta(-g_x, g_y) \in T\} \\ \qquad \text{if } (x,y) + \beta(-g_x,g_y) \in T \text{ for some } \beta \\ \\ -\infty \quad \text{otherwise.} \end{cases} \tag{1}$$

The calculation of the directional technology distance function is depicted in Figure 1.

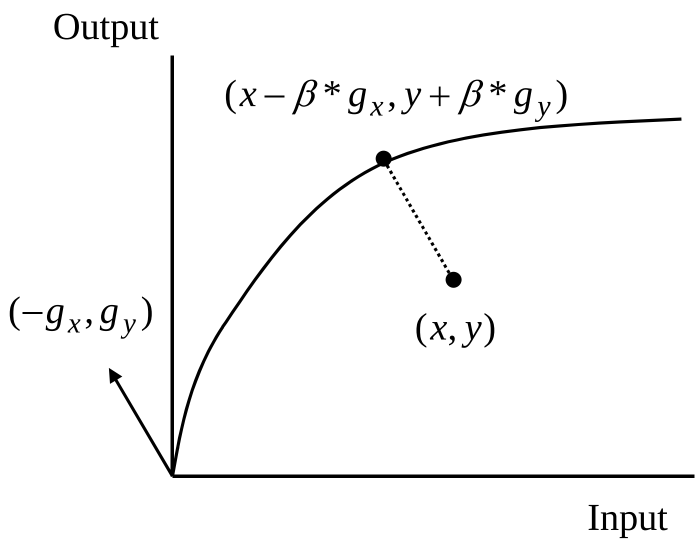

Figure 1

where $\beta^* = \vec{D}(x,y;g_x,g_y)$.

There are, of course, many different implicit representations of a multi-output, muliti-input production technology. However, the directional technology distance function is particularly well-suited to the task of providing a measure of technical efficiency in the full input-output space. To see this consider some of the competing alternative measures.

The hyperbolic measure, proposed by Färe, Grosskopf, and Lovell (1985), is given by

$$F_g(x,y) = \min\left\{\lambda : \left(\lambda x, \frac{y}{\lambda}\right) \in T\right\}.$$

The calculation of this hyperbolic measure is depicted in Figure 2.

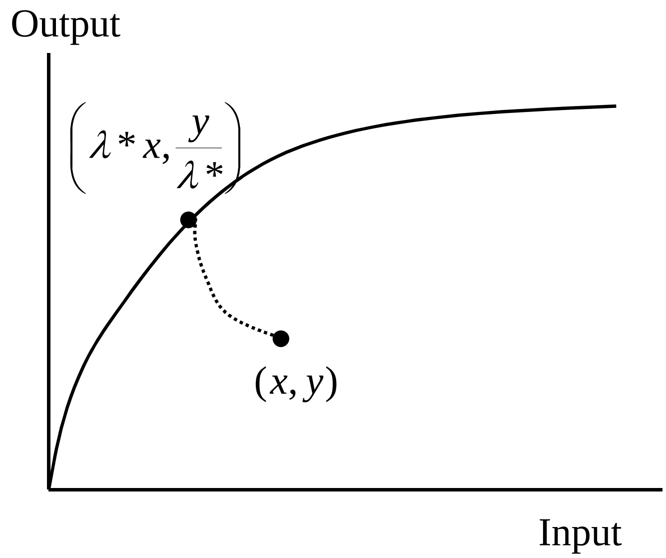

Figure 2

where $\lambda^* = F_g(x, y)$. It is possible to give this measure an economic interpretation but this is done at the expense of assuming constant returns to scale. For the details see Färe, Grosskopf, and Zaim (2002).

Another possibility is the radial measure given by

$$F_R(x, y) = \max \{\delta : (\delta x, \delta y) \in T\} .$$

The calculation of this measure is depicted in Figure 3.

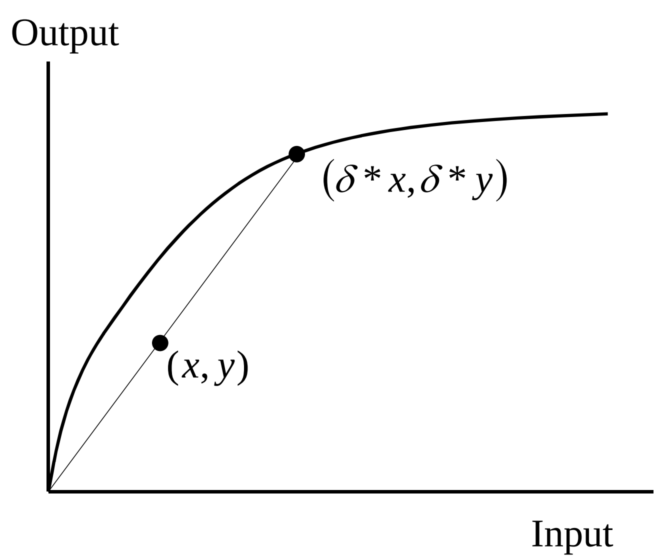

Figure 3

However, this measure could produce very large values (high inefficiency scores) even when (x, y) is very close to the frontier. Moreover, this measure completely breaks down under constant returns to scale.

Lemma 2.2 in Chambers, Chung, and Färe (1998) establishes that A1 - A5 imply the following properties:

D1. Translation Property

$$\vec{D}(x - \alpha g_x, y + \alpha g_y; g_x, g_y) = \vec{D}(x, y; g_x, g_y) - \alpha \text{ for all } \alpha \in R$$

D2. g-Homogeneity of Degree Minus One

$$\vec{D}(x, y; \lambda g_x, \lambda g_y) = \lambda^{-1} \vec{D}(x, y; g_x, g_y), \lambda > 0$$

D3. Input Monotonicity

$$x' \geq x \Rightarrow \vec{D}(x', y; g_x, g_y) \geq \vec{D}(x, y; g_x, g_y)$$

D4. Output Monotonicity

$$y' \geq y \Rightarrow \vec{D}(x, y'; g_x, g_y) \leq \vec{D}(x, y; g_x, g_y)$$

D5. Concavity

$$\vec{D}(x, y; g_x, g_y) \text{ is concave in } (x, y)$$

2 Derivative Properties and Econometric Modelling

An econometric model of the directional technology distance function should impose properties D1 - D5 listed above. This is conveniently accomplished by imposing the restrictions on the first and second derivatives of $\vec{D}(x, y; g_x, g_y)$ that are implied by D1 - D5. These derivative conditions are given in the following lemma.

Lemma 1: Assume that $\vec{D}(x, y; g_x, g_y)$ is twice continuously differentiable. Then D1 - D5 imply that:

DD1. Translation Property

$$\nabla_x \vec{D}(x, y; g_x, g_y) g_x - \nabla_y \vec{D}(x, y; g_x, g_y) g_y = 1$$

DD2. g-Homogeneity of Degree Minus One

$$\nabla_{g_x} \vec{D}(x, y; g_x, g_y) g_x + \nabla_{g_y} \vec{D}(x, y; g_x, g_y) g_y = -\vec{D}(x, y; g_x, g_y)$$

DD3. Input Monotonicity

$$\nabla_x \vec{D}(x, y; g_x, g_y) \geq 0$$

DD4. Output Monotonicity

$$\nabla_y \vec{D}(x, y; g_x, g_y) \leq 0$$

DD5. Concavity

$$H_{\vec{D}} \text{ is negative semidefinite}$$

DD6. Symmetry

$$H_{\vec{D}} \text{ is symmetric}$$

where

$$H_{\vec{D}} = \begin{bmatrix} \nabla_{xx}\vec{D}(x, y; g_x, g_y) & \nabla_{xy}\vec{D}(x, y; g_x, g_y) \\ \nabla_{yx}\vec{D}(x, y; g_x, g_y) & \nabla_{yy}\vec{D}(x, y; g_x, g_y) \end{bmatrix}$$

is the Hessian matrix of $\vec{D}$.

Proof: Differentiating (D1) with respect to α we get

$$-\nabla_x \vec{D}(x - \alpha g_x, y + \alpha g_y; g_x, g_y)g_x + \nabla_y \vec{D}(x - \alpha g_x, y + \alpha g_y; g_x, g_y)g_y = -1.$$

Set α equal to zero and multiply by -1 to get DD1:

$$\nabla_x \vec{D}(x, y; g_x, g_y)g_x - \nabla_y \vec{D}(x, y; g_x, g_y)g_y = 1.$$

D2. says that the directional technology distance function is homogeneous of degree minus one in (g_x, g_y). DD2 follows by Euler's Theorem. DD3 and DD4 follow directly from the monotonicity conditions, D3 and D4, respectively. DD5 follows directly from the concavity of $\vec{D}(x, y; g_x, g_y)$ in (x, y) and DD6. follows from Young's Theorem. QED

Before concluding this section there is one more interesting property to explore. The profit function is defined as

$$\Pi(p, w) = \max_{x,y} \{py - wx : (x, y) \in T\} \tag{2}$$

$$= \max_{x,y} \left\{py - wx : \vec{D}(x, y; g_x, g_y) \geq 0\right\} \tag{3}$$

since

$$(x, y) \in T \Leftrightarrow \vec{D}(x, y; g_x, g_y) \geq 0. \tag{4}$$

Because of (1) and (4) we can write

$$(x, y) \in T \Leftrightarrow (x - \vec{D}(x, y; g_x, g_y)g_x, y + \vec{D}(x, y; g_x, g_y)g_y) \in T,$$

by the free disposability assumption. Thus, profit may be defined by the unconstrained maximization problem:

$$\begin{aligned}\Pi(p,w) &= \max_{x,y}\left\{p\left(y+\vec{D}(x,y;g_x,g_y)g_y\right) - w\left(x-\vec{D}(x,y;g_x,g_y)g_x\right)\right\} \\ &= \max_{x,y}\left\{py - wx + \vec{D}(x,y;g_x,g_y)\left(pg_y+wg_x\right)\right\}\end{aligned}$$

The first order conditions are:

$$\begin{aligned}-w + \nabla_x\vec{D}(x,y;g_x,g_y)\left(pg_y+wg_x\right) &= 0 \\ p + \nabla_y\vec{D}(x,y;g_x,g_y)\left(pg_y+wg_x\right) &= 0\end{aligned}$$

or

$$\frac{w}{pg_y+wg_x} = \nabla_x\vec{D}(x,y;g_x,g_y) \tag{5}$$

$$\frac{p}{pg_y+wg_x} = -\nabla_y\vec{D}(x,y;g_x,g_y) \tag{6}$$

These are the inverse supply and demand functions. Prices (w,p) are normalized by the number, $pg_y + wg_x$.

While this approach is efficient it does not provide an economic interpretation of the term, $pg_y + wg_x$. To provide such an interpretation we turn to a more traditional treatment of the profit maximization problem. Write the Lagrangian function for (3) as

$$L = py - wx + \lambda\vec{D}(x,y;g_x,g_y)$$

First order conditions are:

$$L_x = -w + \lambda\nabla_x\vec{D}(x,y;g_x,g_y) = 0 \Rightarrow \nabla_x\vec{D}(x,y;g_x,g_y) = \frac{w}{\lambda} > 0$$

$$L_y = p + \lambda\nabla_y\vec{D}(x,y;g_x,g_y) = 0 \Rightarrow \nabla_y\vec{D}(x,y;g_x,g_y) = \frac{-p}{\lambda} < 0 \tag{7}$$

or

$$wg_x = \lambda\nabla_x\vec{D}(x,y;g_x,g_y)g_x \tag{8}$$

$$pg_y = -\lambda\nabla_y\vec{D}(x,y;g_x,g_y)g_y \tag{9}$$

Multiplying (DD1) by λ we get:

$$\lambda\nabla_x\vec{D}(x,y;g_x,g_y)g_x - \lambda\nabla_y\vec{D}(x,y;g_x,g_y)g_y = \lambda$$

thus, adding (8) and (9) we get:

$$\begin{aligned}pg_y + wg_x &= \lambda\nabla_x\vec{D}(x,y;g_x,g_y)g_x - \lambda\nabla_y\vec{D}(x,y;g_x,g_y)g_y \\ &= \lambda \\ &\Rightarrow \lambda = pg_y + wg_x \end{aligned} \tag{10}$$

Thus, $pg_y + wg_x$ is the optimal value of the Lagrangian multiplier in the profit maximization problem. If the technology is perturbed (improved) by a small value, ε, from

$$T = \left\{ (x, y) : \vec{D}(x, y; g_x, g_y) \geq 0 \right\}$$

to

$$T' = \left\{ (x, y) : \vec{D}(x, y; g_x, g_y) + \varepsilon \geq 0 \right\}$$

then the firm's profit will rise and $\frac{\partial \Pi(p,w)}{\partial \varepsilon} = pg_y + wg_x$.[1]

Putting (10) into (7) and rearranging we get

$$\begin{aligned} \frac{w}{pg_y + wg_x} &= \nabla_x \vec{D}(x, y; g_x, g_y) \\ \frac{p}{pg_y + wg_x} &= -\nabla_y \vec{D}(x, y; g_x, g_y) \end{aligned}$$

which, of course, is the same result as (5) and (6).

3 Comparative Statics

In this section we show how comparative static derivatives of the input demand and the output supply functions may be expressed as functions of the first and second order derivatives of the directional technology distance function. Rearranging (5) and (6), we get

$$\nabla_x \vec{D}(x, y) \, (pg_y + wg_x) = w \tag{11}$$

$$\nabla_y \vec{D}(x, y) \, (pg_y + wg_x) = -p \tag{12}$$

First, differentiate (11) and (12) with respect to the input price vector, w.

$$\begin{aligned} \nabla_x \vec{D}(x, y) g_x + \left[\nabla_{xx} \vec{D}(x, y) \frac{\partial x}{\partial w} + \nabla_{xy} \vec{D}(x, y) \frac{\partial y}{\partial w} \right] (pg_y + wg_x) = 1 \\ \nabla_y \vec{D}(x, y) g_x + \left[\nabla_{yx} \vec{D}(x, y) \frac{\partial x}{\partial w} + \nabla_{yy} \vec{D}(x, y) \frac{\partial y}{\partial w} \right] (pg_y + wg_x) = 0 \end{aligned}$$

and write the result, rearranged, n matrix notation,

$$\begin{bmatrix} \nabla_{xx} \vec{D}(x, y) & \nabla_{xy} \vec{D}(x, y) \\ \nabla_{yx} \vec{D}(x, y) & \nabla_{yy} \vec{D}(x, y) \end{bmatrix} \begin{bmatrix} \dfrac{\partial x}{\partial w} \\ \\ \dfrac{\partial y}{\partial w} \end{bmatrix} = \frac{1}{pg_y + wg_x} \begin{bmatrix} 1 - \nabla_x \vec{D}(x, y) g_x \\ -\nabla_y \vec{D}(x, y) g_x \end{bmatrix}.$$

[1] It is also possible to infer this result from the proof in the Appendix of Chambers, Chung, and Färe (1998).

Next, differentiate (11) and (12) with respect to output prices, p.

$$\nabla_x \vec{D}(x,y)g_y + \left[\nabla_{xx}\vec{D}(x,y)\frac{\partial x}{\partial p} + \nabla_{xy}\vec{D}(x,y)\frac{\partial y}{\partial p}\right](pg_y + wg_x) = 0$$
$$\nabla_y \vec{D}(x,y)g_y + \left[\nabla_{yx}\vec{D}(x,y)\frac{\partial x}{\partial p} + \nabla_{yy}\vec{D}(x,y)\frac{\partial y}{\partial p}\right](pg_y + wg_x) = -1.$$

Rearrange and write in matrix notation.

$$\begin{bmatrix} \nabla_{xx}\vec{D}(x,y) & \nabla_{xy}\vec{D}(x,y) \\ \nabla_{yx}\vec{D}(x,y) & \nabla_{yy}\vec{D}(x,y) \end{bmatrix} \begin{bmatrix} \dfrac{\partial x}{\partial p} \\ \dfrac{\partial y}{\partial p} \end{bmatrix} = \frac{1}{pg_y + wg_x} \begin{bmatrix} -\nabla_x \vec{D}(x,y)g_y \\ -1 - \nabla_y \vec{D}(x,y)g_y \end{bmatrix}.$$

$$\begin{bmatrix} \nabla_{xx}\vec{D}(x,y) & \nabla_{xy}\vec{D}(x,y) \\ \nabla_{yx}\vec{D}(x,y) & \nabla_{yy}\vec{D}(x,y) \end{bmatrix} \begin{bmatrix} \dfrac{\partial x}{\partial w} & \dfrac{\partial x}{\partial p} \\ \dfrac{\partial y}{\partial w} & \dfrac{\partial y}{\partial p} \end{bmatrix}$$

$$= \frac{1}{pg_y + wg_x} \begin{bmatrix} 1 - \nabla_x \vec{D}(x,y)g_x & -\nabla_x \vec{D}(x,y)g_y \\ -\nabla_y \vec{D}(x,y)g_x & -1 - \nabla_y \vec{D}(x,y)g_y \end{bmatrix}$$

$$= \frac{1}{pg_y + wg_x} \begin{bmatrix} -\nabla_y \vec{D}(x,y)g_y & -\nabla_x \vec{D}(x,y)g_y \\ -\nabla_y \vec{D}(x,y)g_x & -\nabla_x \vec{D}(x,y)g_x \end{bmatrix} \qquad \text{(using DD1.)}$$

$$= \frac{-1}{pg_y + wg_x} \begin{bmatrix} \nabla_y \vec{D}(x,y)g_y & \nabla_x \vec{D}(x,y)g_y \\ \nabla_y \vec{D}(x,y)g_x & \nabla_x \vec{D}(x,y)g_x \end{bmatrix}$$

Thus, the matrix of comparative static derivatives of the input demand and the output supply functions can be found above after we invert the Hessian matrix of the directional technology distance function. We get,

$$\begin{bmatrix} -\nabla_{ww}\Pi(p,w) & -\nabla_{wp}\Pi(p,w) \\ \nabla_{pw}\Pi(p,w) & \nabla_{pp}\Pi(p,w) \end{bmatrix} = \begin{bmatrix} \dfrac{\partial x}{\partial w} & \dfrac{\partial x}{\partial p} \\ \dfrac{\partial y}{\partial w} & \dfrac{\partial y}{\partial p} \end{bmatrix}$$

$$= \frac{-1}{pg_y + wg_x} \begin{bmatrix} \nabla_{xx}\vec{D}(x,y) & \nabla_{xy}\vec{D}(x,y) \\ \nabla_{yx}\vec{D}(x,y) & \nabla_{yy}\vec{D}(x,y) \end{bmatrix}^{-1} \begin{bmatrix} \nabla_y \vec{D}(x,y)g_y & \nabla_x \vec{D}(x,y)g_y \\ \nabla_y \vec{D}(x,y)g_x & \nabla_x \vec{D}(x,y)g_x \end{bmatrix}$$

4 Functional Forms

Econometric estimation of a directional distance function requires the choice of a functional form. We will begin our discussion of this choice for the case in which $(g_x, g_y) = \left(1^N, 1^M\right)$. Chambers (1978) suggested two different functional forms in this case, namely, the *logarithmic-transcendental* and the *quadratic.* These suggestions were later validated in a paper by Färe and Lundberg (2004). They sought functional forms that satisfy the translation property, D.1, and that have a second order Taylor series approximation interpretation. A function of n variables, F, has a second order Taylor series approximation interpretation if there are real constants, $a_i, b_{jk}, i, j, k = 1, ..., n$ and real-valued functions ϕ and h such that

$$\phi\left(F(z)\right) = \sum_{i=1}^{n} a_i h(z_i) + \sum_{j=1}^{n}\sum_{k=1}^{n} b_{jk} h(z_j) h(z_k), \tag{13}$$

where it is assumed, without loss of generality, that $b_{jk} = b_{kj},\ \ j, k = 1, ..., n$. See Lau (1977) and Blackorby, Primont, and Russell (1978, pp. 290-296) for a further discussion.

The two functional forms that they found are the quadratic

$$T\left(z\right) = \sum_{i=1}^{N+M} a_i z_i + \sum_{j=1}^{N+M}\sum_{k=1}^{N+M} b_{jk} z_j z_k, \tag{14}$$

where $n = N + M, z = (x, y)$ and $T(z) = T(x, y) = D(x, y; 1^N, 1^M)$, and what we will call the *transcendental-exponential*

$$\begin{aligned} &T\left(x, y\right) \\ &= \frac{1}{2\lambda} \ln \Bigg\{ \sum_{i=1}^{N}\sum_{j=1}^{N} a_{ij} \exp\left(\lambda x_i\right) \exp\left(\lambda x_j\right) + \sum_{k=1}^{M}\sum_{\ell=1}^{M} b_{k\ell} \exp\left(-\lambda y_k\right) \exp\left(-\lambda y_\ell\right) \\ &\qquad + \sum_{i=1}^{N}\sum_{k=1}^{M} c_{ik} \exp\left(\lambda x_i\right) \exp\left(-\lambda y_k\right) \Bigg\}. \end{aligned} \tag{15}$$

The quadratic (14) is linear in the parameters and can be readily estimated. The transcendental-exponential (15) can be linearized in all of the parameters except for

λ. Setting $\lambda = \frac{1}{2}$ and exponentiating both sides of (15) yields

$$\begin{aligned}
&\exp(T(x,y)) \\
&= \sum_{i=1}^{N}\sum_{j=1}^{N} a_{ij} \exp\left(\frac{x_i}{2}\right) \exp\left(\frac{x_j}{2}\right) + \sum_{k=1}^{M}\sum_{\ell=1}^{M} b_{k\ell} \exp\left(-\frac{y_k}{2}\right) \exp\left(-\frac{y_\ell}{2}\right) \\
&\quad + \sum_{i=1}^{N}\sum_{k=1}^{M} c_{ik} \exp\left(\frac{x_i}{2}\right) \exp\left(-\frac{y_k}{2}\right). \qquad (16)
\end{aligned}$$

Equations (14) and (16) are the functional forms first suggested by Chambers (1998).

It can be verified that both (15) and (16) automatically satisfy the translation property. The quadratic functional form satisfies the translation property if the following linear parametric restrictions are imposed.

$$\sum_{i=1}^{N} a_i - \sum_{k=1}^{M} a_k = 1, \quad \sum_{j=1}^{M} b_{jk} - \sum_{j=1}^{N} b_{jk} = \sum_{k=1}^{M} b_{jk} - \sum_{k=1}^{N} b_{jk} = 0 \qquad (17)$$

The restrictions in (17) will be derived in a more general setting below. Examples of the use of the quadratic functional form for directional distance functions include Färe, Grosskopf, and Weber (2001) and Färe, Grosskopf, Noh, and Weber (2005).[2]

We now consider any given direction vector, (g_x, g_y). Retaining our notation, $z = (x, y)$, and letting $g = (-g_x, g_y)$, the directional distance function is defined as

$$\vec{\mathcal{D}}(z; g) = \sup_{\beta} \{\beta : z + \beta g \in T\},$$

i.e., $\vec{\mathcal{D}}(z; g) = \vec{\mathcal{D}}(x, y; -g_x, g_y) = \vec{D}(x, y; g_x, g_y)$. In terms of $\vec{\mathcal{D}}$ the translation property is again established by

$$\begin{aligned}
\vec{\mathcal{D}}(z + \alpha g; g) &= \sup_{\beta} \{\beta : z + \alpha g + \beta g \in T\} \\
&= \sup_{\beta} \{\beta : z + (\alpha + \beta) g \in T\} \\
&= -\alpha + \sup_{\alpha+\beta} \{\alpha + \beta : z + (\alpha + \beta) g \in T\} \\
&= \vec{\mathcal{D}}(z; g) - \alpha.
\end{aligned}$$

Again, we seek functional forms that meet the Färe-Lundberg conditions, namely, 1) they satisfy the translation property and 2) they have a second-order Taylor series approximation interpretation. Any functional form that does satisfy the Färe-Lundberg conditions for any direction (g_x, g_y) must also satisfy the Färe-Lundberg

[2] Actually these two papers use the directional output distance function. This entails setting $g_x = 0^N$.

conditions for the direction $(g_x, g_y) = \left(1^N, 1^M\right)$. Hence, the only candidates for such functional forms are the quadratic (14) and the transcendental exponential (15) function forms.

For the rest of this section we will be content to show that the quadratic functional form still "works" for any direction vector (g_x, g_y). We will impose the translation property on the quadratic functional form and thereby derive the restrictions imposed by the translation property. For the quadratic functional form.

$$\sum_{i=1}^{n} a_i z_i + \sum_{j=1}^{n} \sum_{k=1}^{n} b_{jk} z_j z_k$$

we want the following to hold identically for all α and for all z.

$$\begin{aligned}
&\sum_i a_i (z_i + \alpha g_i) + \sum_j \sum_k b_{jk} (z_j + \alpha g_j)(z_k + \alpha g_k) \\
&= \sum_i a_i (z_i + \alpha g_i) + \sum_j \sum_k b_{jk} \left(z_j z_k + \alpha z_j g_k + \alpha z_k g_j + \alpha^2 g_j g_k\right) \\
&= \sum_i a_i z_i + \sum_j \sum_k b_{jk} z_j z_k - \alpha
\end{aligned}$$

Cancelling common terms and factoring out the $\alpha' s$ we get

$$\alpha \sum_i a_i g_i + \alpha \sum_j \sum_k b_{jk} (z_j g_k + z_k g_j) + \alpha^2 \sum_j \sum_k b_{jk} g_j g_k = -\alpha$$

or

$$\alpha \sum_i a_i g_i + \alpha \sum_j z_j \sum_k b_{jk} g_k + \alpha \sum_k z_k \sum_j b_{jk} g_j + \alpha^2 \sum_j \sum_k b_{jk} g_j g_k = -\alpha$$

Divide both sides by α to get

$$\sum_i a_i g_i + \sum_j z_j \sum_k b_{jk} g_k + \sum_k z_k \sum_j b_{jk} g_j + \alpha \sum_j \sum_k b_{jk} g_j g_k = -1 \qquad (18)$$

Differentiate (18) with respect to z_l to get

$$\sum_k b_{jk} g_k + \sum_j b_{jk} g_j = 0$$

or

$$\sum_k b_{kj} g_k + \sum_j b_{jk} g_j = 0,$$

using symmetry. Hence

$$2 \sum_j b_{jk} g_j = 0 \Rightarrow \sum_j b_{jk} g_j = 0.$$

We conclude that

$$\sum_k b_{kj} g_k = \sum_j b_{jk} g_j = 0 \tag{19}$$

Then (18) and (19) imply that

$$\sum_j \sum_k b_{jk} g_j g_k = 0 \text{ and hence } \sum_i a_i g_i = -1.$$

We summarize these restrictions below.

$$\sum_i a_i g_i = -1, \quad \sum_j b_{jk} g_j = \sum_k b_{jk} g_k = 0 \tag{20}$$

Now, of course, $g = (-g_x, g_y)$. If we let $(g_x, g_y) = \left(1^N, 1^M\right)$ so that $g = \left(-1^N, 1^M\right)$, then (20) becomes

$$-\sum_{i=1}^{N} a_i + \sum_{k=1}^{M} a_k = -1, \quad \sum_{j=1}^{M} b_{jk} - \sum_{j=1}^{N} b_{jk} = \sum_{k=1}^{M} b_{jk} - \sum_{k=1}^{N} b_{jk} = 0,$$

or

$$\sum_{i=1}^{N} a_i - \sum_{k=1}^{M} a_k = 1, \quad \sum_{j=1}^{M} b_{jk} - \sum_{j=1}^{N} b_{jk} = \sum_{k=1}^{M} b_{jk} - \sum_{k=1}^{N} b_{jk} = 0. \tag{21}$$

The conditions in (21) coincide with the conditions in (17).

5 Closing Remarks

In this paper we have established the derivative restrictions on the directional technology distance function that would be useful in econometric work. It was shown that the standard neoclassical comparative static analysis for a competitive firm can be easily handled with the directional technology distance function. Finally, we have briefly surveyed the functional forms that seem to be best suited for econometric estimation. There are, of course, other uses of the directional technology distance function. In addition to the previously cited papers by Färe, Grosskopf, and Weber (2001) and Färe, Grosskopf, Noh, and Weber (2005), Färe and Grosskopf (2000) show, among other things, that the directional technology distance function can be used to model plant capacity. For another example, Färe and Primont (2003) use the directional technology distance function to find conditions under which productivity indicators for each firm in an industry can be aggregated to a productivity indicator for the industry as a whole.

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