Dual Allocative Efficiency Parameters

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ABSTRACT: Estimation of either price or quantity efficiency parameters provides an empirical method for taking into account the presence of allocative inefficiency. We show that the researcher can either a) estimate a system of input demand functions of the form, $x = h(y, k \circ w)$, obtain estimated values of the price efficiency parameters (the $k's$), and then derive the quantity efficiency parameters with a simple calculation or b) estimate a system of shadow price functions of the form, $w = g(y, \delta \circ x)$, obtain estimated values of the quantity efficiency parameters (the $\delta's$), and then derive the price efficiency parameters with a simple calculation.

1 Introduction

Estimation of either price or quantity efficiency parameters provides an empirical method for taking into account the presence of allocative inefficiency. In this paper, we explore the relationship between these two sets of parameters in the duality theory framework of a competitive, cost-minimizing firm.

Suppose we are given an observation consisting of an output vector, $y = (y_1, ..., y_M)$, and input vector, $x^0 = (x^0_1, ..., x^0_N)$, and an input price vector, $w^0 = (w^0_1, ..., w^0_N)$. The choice of x^0 may not be allocatively efficient (and hence, not cost-minimizing) given (y, w^0). To account for this inefficiency one may introduce price allocative efficiency parameters, $k_1, ..., k_N$, with the property that x^0 is allocatively efficient given $(y, k \circ w^0)$, where $k \circ w^0 = (k_1 w^0_1, ..., k_N w^0_N)$. In other words, $k \circ w^0$ is a shadow price vector for x^0. This sort of empirical strategy was used by Lau and Yotopoulos (1971), Atkinson and Halvorsen (1980), and Lovell and Sickles (1983) in the context of profit maximization and by Toda (1976), Atkinson and Halvorsen (1984), and Atkinson and Cornwell (1994) in the context of cost minimization.

Alternatively, one can introduce quantity efficiency parameters, $\delta_1, ..., \delta_N$, with the property that $\delta \circ x^0$ is allocatively efficient given (y, w^0), where $\delta \circ x^0 = (\delta_1 x^0_1, ..., \delta_N x^0_N)$. In other words, w^0 is a shadow price vector for $\delta \circ x^0$. This approach was employed by Atkinson and Primont (2002) and by Atkinson, Färe, and Primont (2003).

In this paper, we answer the following research question. What is the relationship between the price efficiency parameters and the quantity efficiency parameters? It will be shown that the researcher can either a) estimate a system of input demand functions of the form, $x = h(y, k \circ w)$, obtain estimated values of the price efficiency parameters and then derive the quantity efficiency parameters with a simple calculation or b) estimate a system of shadow price functions of the form, $w = g(y, \delta \circ x)$, obtain estimated values of the quantity efficiency parameters and then derive the price efficiency parameters with a simple calculation. These results are applications of the duality theory for cost and input distance functions. Thus, we have another example in which duality theory gives you two for the price of one.

2 Some Duality Theory

The research question posed above would be fairly easy to settle if the system of input demand functions and the system of shadow price functions shared an inverse relationship, i.e., if it were true that

$$x = h(y, w) \text{ if and only if } w = g(y, x).$$

This is not, however, the case since both $h(y, \cdot)$ and $g(y, \cdot)$ are homogeneous of degree zero in the second vector. In the differentiable case this implies that the Jacobian matrix of h and of g is singular and the Implicit Function Theorem does not apply. Deaton (1979) dealt with this problem by using generalized inverses. In this paper, we overcome this difficulty with some simple duality results. This section provides a brief review of the duality theory that is the foundation of the results of this paper. While much of this material is familiar, the theory is restated in a way that makes it immediately applicable to the question posed in this paper. A more thorough and rigorous treatment of the duality theory used here may be found in Färe and Primont (1995).

Input requirement sets are given by $L(y) = \left\{x \in R_+^N : x \text{ can produce } y\right\}$ for each $y \in R_+^M$. The input distance function is defined by $D_i(y, x) = \sup_\lambda \{\lambda : (x/\lambda) \in L(y)\}$. Fix the output vector at y and let the input price vector be some arbitrary vector w^0. The cost function is derived from the input distance function by the following cost minimization problem (hereafter referred to as CMP)

$$C(y, w^0) = \min_x \left\{w^0 x : D_i(y, x) \geq 1\right\} = w^0 x^* = w^0 h(y, w^0) \tag{1}$$

where the solution, $x^* = h(y, w^0)$, is the $N \times 1$ vector of optimal input demands with components:

$$x_n^* = h_n(y, w^0), \qquad n = 1, ..., N. \tag{2}$$

As is well known, $h(y, \cdot)$ is homogeneous of degree zero in the input price vector.

Note that the constraint in (1) will be binding and, thus, $D_i(y, x^*) = 1$. The input distance function can be recovered from the cost function by the following shadow pricing problem (hereafter referred to as the SPP)

$$D_i(y, x^*) = \min_w \{wx^* : C(y, w) \geq 1\} = g(y, x^*)x^*. \tag{3}$$

where the solution, $g(y, x^*)$, is the $N \times 1$ vector of shadow prices with components, $g_n(y, x^*), n = 1, ..., N$. Moreover,

$$C(y, w^0) D_i(y, x^*) = w^0 x^*, \tag{4}$$

since $D_i(y, x^*) = 1$.

Lemma 1 If x^* solves the CMP (1) at prices, w^0, then the normalized price vector $w^0/C(y, w^0)$ solves the SPP (3) at quantities, x^*, i.e.,

$$\frac{w_n^0}{C(y, w^0)} = g_n(y, x^*), \qquad n = 1, ..., N. \tag{5}$$

Proof: The normalized price vector is clearly feasible since $C\left(y, w^0/C(y, w^0)\right) = 1$.

Moreover, (4) implies that

$$\frac{w^0 x^*}{C(y, w^0)} = D_i(y, x^*),$$

and therefore $w^0/C(y, w^0)$ also yields the optimal value of (3).■

Again fix the output vector at y and let the input vector be some arbitrary vector x^0. The input distance function is derived from the cost function by the shadow pricing problem (SPP)

$$D_i(y, x^0) = \min_w \left\{ wx^0 : C(y, w) \geq 1 \right\} = w^* x^0 = g(y, x^0) x^0, \tag{6}$$

where the solution, $w^* = g(y, x^0)$, is the $N \times 1$ vector of shadow prices with components

$$w_n^* = g_n(y, x^0), \qquad n = 1, ..., N. \tag{7}$$

The shadow price function, $g(y, \cdot)$, is homogeneous of degree zero in the input quantity vector.

Note that the constraint in (6) will be binding and, thus, $C(y, w^*) = 1$. The cost function can be recovered from the input distance function by the cost-minimization problem (CMP)

$$C(y, w^*) = \min_x \left\{ w^* x : D_i(y, x) \geq 1 \right\} = w^* h(y, w^*) \tag{8}$$

where the solution, $h(y, w^*)$, is the $N \times 1$ vector of optimal input demands with components, $h_n(y, w^*), n = 1, ..., N$. Moreover,

$$C(y, w^*) D_i(y, x^0) = w^* x^0 \tag{9}$$

since $C(y, w^*) = 1$.

Lemma 2 If w^* solves the SSP (6) at quantities, x^0, then the normalized input vector, $x^0/D_i(y, x^0)$, solves the CMP (8) at prices, w^*, i.e.,

$$\frac{x_n^0}{D_i(y, x^0)} = h_n(y, w^*), \qquad n = 1, ..., N. \tag{10}$$

Proof: The normalized input vector is feasible since $D_i(y, x^0/D_i(y, x^0)) = 1$. Moreover, (9) implies that

$$w^* \frac{x^0}{D_i(y, x^0)} = C(y, w^*),$$

and therefore $x^0/D_i(y, x^0)$ also yields the optimal value of (8).■

3 Efficiency Parameters

Suppose a data point is given by (w^0, x^0, y). Since the observed, normalized input vector, $x^0/D_i(y, x^0)$, may differ from the optimal input vector, x^*, we introduce N quantity efficiency parameters defined by

$$(11) \qquad \delta_n = \frac{x_n^*}{x_n^0/D_i(y, x^0)}, \qquad n = 1, ..., N.$$

Then

$$(12) \qquad x_n^* = \frac{\delta_n x_n^0}{D_i(y, x^0)} \qquad n = 1, ..., N.$$

or

$$(13) \qquad x^* = \frac{\delta \circ x^0}{D_i(y, x^0)} = \left(\frac{\delta_1 x_1^0}{D_i(y, x^0)}, ..., \frac{\delta_N x_N^0}{D_i(y, x^0)} \right),$$

where $\delta \circ x^0$ is the Hadamard product.

Alternatively, we may account for the difference between the normalized input price vector, $w^0/C(y, w^0)$, and the optimal shadow price vector, w^*, by introducing N price efficiency parameters defined by

$$(14) \qquad k_n = \frac{w_n^*}{w_n^0/C(y, w^0)}, \qquad n = 1, ..., N.$$

Then we have

$$(15) \qquad w_n^* = \frac{k_n w_n^0}{C(y, w^0)}, \qquad n = 1, ..., N,$$

or

$$(16) \qquad w^* = \frac{k \circ w^0}{C(y, w^0)} = \left(\frac{k_1 w_1^0}{C(y, w^0)}, ..., \frac{k_N w_N^0}{C(y, w^0)} \right),$$

where $k \circ w^0$ is the Hadamard product.

Lemmas 1 and 2 and the definitions of the price and quantity allocative efficiency parameters are illustrated in the following four-quadrant diagram (Figure 1).

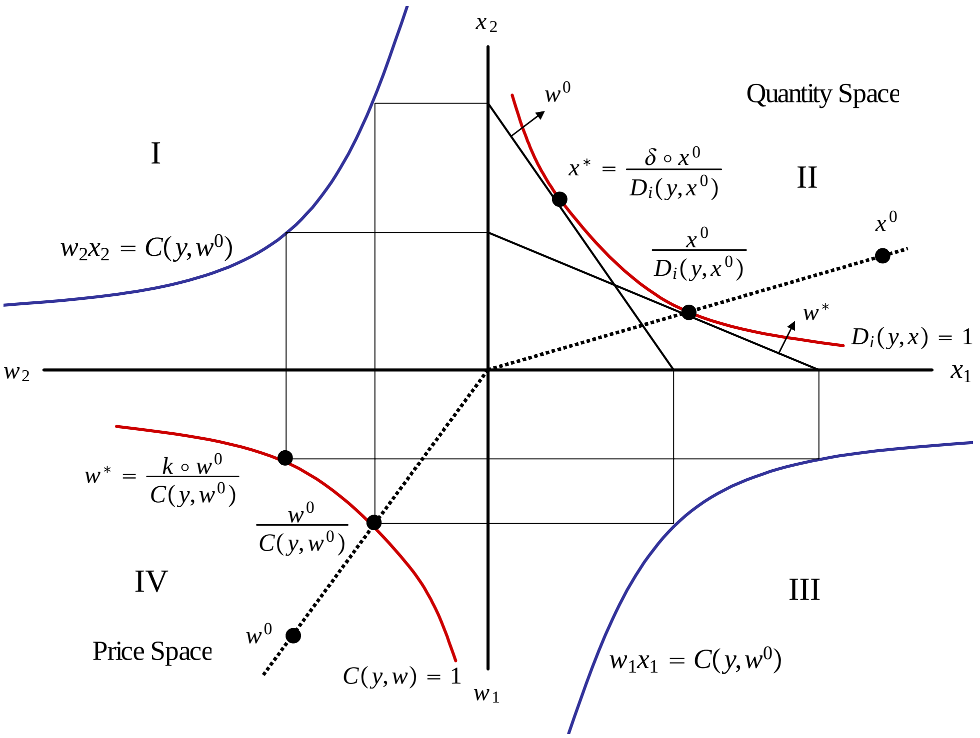

Figure 1

An isoquant for the output vector y is given in the quantity space and is defined by the equation, $D_i(y,x) = 1$. It is the set of input vectors that can just produce y. The corresponding isoquant for output vector y in price space is defined by the equation, $C(y,w) = 1$. It is the set of input price vectors at which the firm can hire the least-cost combination of inputs that can produce y for a cost of one dollar.

Begin with x^0 in quadrant II (quantity space). Deflation by $D_i(y,x^0)$ results in the point $x^0/D_i(y,x^0)$ whose shadow price vector is w^*. Hadamard multiplication by δ results in $\delta \circ x^0/D_i(y,x^0)$ whose shadow price vector is w^0. In quadrant III (I) draw a rectangular hyperbola given by the equation $w_1 x_1 = C(y,w^0)$ $(w_2 x_2 = C(y,w^0))$.

The endpoints of the isocost line through the point $x^* = \delta \circ x^0/D_i(y,x^0)$ are given by $C(y,w^0)/w_1^0$ and $C(y,w^0)/w_2^0$. If these two points are projected onto the corresponding rectangular hyperbola and then projected into price space these two ratios are "flipped over" and we end up at the point $w^0/C(y,w^0)$. Thus, the input vector, x^*, solves the CMP at w^0 and $w^0/C(y,w^0)$ solves the SSP at x^*. This is just Lemma 1.

Similarly, the endpoints of the isocost line through the point $x^0/D_i(y,x^0)$ are given by $C(y,w^*)/w_1^*$ and $C(y,w^*)/w_2^*$. If these two points are projected onto the corresponding rectangular hyperbola and then projected into price space these two ratios are "flipped over" and we end up at the point $w^*/C(y,w^*) = w^*$ since $C(y,w^*) = 1$. Thus, the input vector, $x^0/D_i(y,x^0)$, solves the CMP at w^* and w^* solves the SPP at quantities $x^0/D_i(y,x^0)$. This is a result of Lemma 2 and the zero-degree homogeneity of the shadow price function in the input quantities.

In an analogous way, begin with w^0 in quadrant IV (price space) depicted in Figure 2. Deflation by $C(y, w^0)$ results in the point $w^0/C(y, w^0)$ whose supporting hyperplane is defined by x^*. Hadamard multiplication by k results in $k \circ w^0/C(y, w^0)$ whose supporting hyperplane is given by x^0. In quadrant III (I) draw a rectangular hyperbola given by the equation $w_1x_1 = D_i(y, x^0)$ $(w_2x_2 = D_i(y, x^0))$.

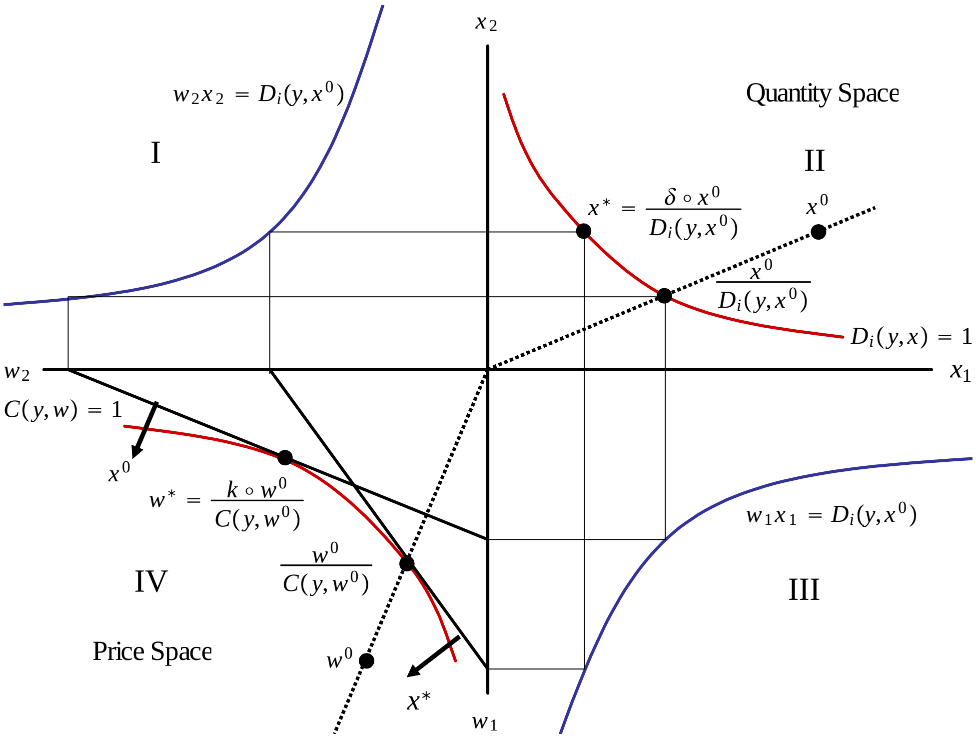

Figure 2

The endpoints of the isocost line through the point $w^* = k \circ w^0/C(y, w^0)$ are given by $D_i(y, x^0)/x_1^0$ and $D_i(y, x^0)/x_2^0$. If these two points are projected onto the corresponding rectangular hyperbola and then projected into quantity space these two ratios are flipped over and we end up at the point $x^0/D_i(y, x^0)$. Thus, the input price vector w^* solves the SSP at quantities x^0 and the input vector $x^0/D_i(y, x^0)$ solves the CMP at w^*. This is just Lemma 2.

Similarly, the endpoints of the isocost line through the point $w^0/C(y, w^0)$ are given by $D_i(y, x^*)/x_1^*$ and $D_i(y, x^*)/x_2^*$. If these two points are projected onto the corresponding rectangular hyperbola and then projected into quantity space they are flipped over and we end up at the point x^*. Thus, the input price vector $w^0/C(y, w^0)$ solves the SSP at quantities x^* and x^* solves the CMP at prices $w^0/C(y, w^0)$. This is a result of Lemma 1 and zero-degree homogeneity of the input demand functions in the input prices.

In both Figures 1 and 2, Hadamard multiplication by δ induces a movement along the isoquant in quantity space and corresponds to a dual movement along the isoquant in price space induced by Hadamard multiplication by k. This graphical relationship is further characterized in our main result.

Theorem The price and quantity allocative efficiency parameters defined in (14) and (11), respectively, satisfy the following set of equations.

$$k_n = \frac{g_n(y, x^0)}{g_n\left(y, \delta \circ x^0\right)} \qquad n = 1, ..., N. \tag{17}$$

and

$$\delta_n = \frac{h_n(y, w^0)}{h_n\left(y, k \circ w^0\right)} \qquad n = 1, ..., N. \tag{18}$$

Proof: Substituting (12) into (2), (13) into (5), (15) into (7), and (16) into (10), we get

$$\frac{\delta_n x_n^0}{D_i(y, x^0)} = h_n(y, w^0), \qquad n = 1, ..., N. \tag{19}$$

$$\frac{w_n^0}{C(y, w^0)} = g_n\left(y, \frac{\delta \circ x^0}{D_i(y, x^0)}\right), \qquad n = 1, ..., N. \tag{20}$$

$$\frac{k_n w_n^0}{C(y, w^0)} = g_n(y, x^0), \qquad n = 1, ..., N. \tag{21}$$

and

$$\frac{x_n^0}{D_i(y, x^0)} = h_n\left(y, \frac{k \circ w^0}{C(y, w^0)}\right), \qquad n = 1, ..., N. \tag{22}$$

Then, dividing (21) by (20) we get

$$k_n = \frac{g_n(y, x^0)}{g_n\left(y, \dfrac{\delta \circ x^0}{D_i(y, x^0)}\right)} \qquad n = 1, ..., N \tag{23}$$

and dividing (19) by (22) we get

$$\delta_n = \frac{h_n(y, w^0)}{h_n\left(y, \dfrac{k \circ w^0}{C(y, w^0)}\right)} \qquad n = 1, ..., N \tag{24}$$

Since each $g_n(y,\cdot)$ and $h_n(y,\cdot)$ is homogeneous of degree zero in the second vector we may rewrite (23) and (24) as (17) and (18), respectively.■

In general, the k_n's will be functions of the output vector, the observed input vector, and the vector of the δ_n's; the δ_n's will be functions of the output vector, the observed input price vector, and the vector of the k_n's. The next section presents a simple Cobb-Douglas example for which (17) and (18) take on an extremely simple form.

4 The Cobb-Douglas Example

For the Cobb-Douglas production function

$$y = A\prod_{n=1}^{N} x_n^{a_n}, \qquad a_n > 0, n = 1, ..., N,$$

let $a = \sum_{n=1}^{N} a_n$. The cost function has the form

$$C(y,w) = a\prod_{j=1}^{N}\left(\frac{w_j}{a_j}\right)^{a_j/a}\left(\frac{y}{A}\right)^{1/a}$$

and the input demands are

$$x_n^* = h_n(y,w) = \frac{a_n}{w_n}\prod_{j=1}^{N}\left(\frac{w_j}{a_j}\right)^{a_j/a}\left(\frac{y}{A}\right)^{1/a}, \qquad n = 1, ..., N$$

Hence, using (18) we get

$$\begin{aligned}\delta_n &= \frac{\dfrac{a_n}{w_n}\prod\limits_{j=1}^{N}\left(\dfrac{w_j}{a_j}\right)^{a_j/a}\left(\dfrac{y}{A}\right)^{1/a}}{\dfrac{a_n}{k_n w_n}\prod\limits_{j=1}^{N}\left(\dfrac{k_j w_j}{a_j}\right)^{a_j/a}\left(\dfrac{y}{A}\right)^{1/a}} \\ &= \frac{k_n}{\prod\limits_{j=1}^{N} k_j^{a_j/a}}, \qquad n = 1, ..., N \qquad (25)\end{aligned}$$

Thus, in the Cobb-Douglas case, the quantity efficiency parameters depend only on the price efficiency parameters and are independent of input prices and outputs. Also note that if we raise both sides of (25) to the power a_n/a and multiply over n

we get

$$\prod_{n=1}^{N} \delta_n^{a_n/a} = \frac{\prod\limits_{n=1}^{N} k_n^{a_n/a}}{\prod\limits_{j=1}^{N} k_j^{a_j/a}} = 1. \tag{26}$$

The input distance function has the form

$$D_i(y,x) = \prod_{j=1}^{N} x_j^{a_j/a} \left(\frac{A}{y}\right)^{1/a}$$

and the inverse input demand functions are

$$w_n^* = g_n(y,x) = \frac{a_n/a}{x_n} \prod_{j=1}^{N} x_j^{a_j/a} \left(\frac{A}{y}\right)^{1/a}, n = 1, ..., N.$$

Hence, using (17)

$$\begin{aligned} k_n &= \frac{\frac{a_n/a}{x_n} \prod\limits_{j=1}^{N} x_j^{a_j/a} \left(\frac{A}{y}\right)^{1/a}}{\frac{a_n/a}{\delta_n x_n} \prod\limits_{j=1}^{N} (\delta_j x_j)^{a_j/a} \left(\frac{A}{y}\right)^{1/a}} \\ &= \frac{\delta_n}{\prod\limits_{j=1}^{N} \delta_j^{a_j/a}} \end{aligned} \tag{27}$$

If we raise each side of (27) to the power a_n/a and multiply over n we get

$$\prod_{n=1}^{N} k_n^{a_n/a} = \frac{\prod\limits_{n=1}^{N} \delta_n^{a_n/a}}{\prod\limits_{j=1}^{N} \delta_j^{a_j/a}} = 1. \tag{28}$$

Now, from (26) and (27) we conclude that

$$k_n = \delta_n, n = 1, ..., N.$$

5 Closing Remarks

The answer to the question posed in the introduction is contained in equations (17) and (18). If one has estimated the input demand system along with the price efficiency parameters then the nth quantity efficiency parameter is easily calculated as the nth input demand evaluated at the observed input price vector divided by the nth input demand evaluated at the shadow input price vector If one has, instead, estimated the inverse input demand system along with the quantity efficiency parameters then the nth price efficiency parameter is easily calculated as the nth inverse input demand evaluated at the observed input vector divided by the nth inverse input demand evaluated at the optimal input vector. Of course, one can also estimate both systems. It might be interesting to compare efficiency parameters that are directly estimated to those that are found by the above calculations.

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