# The Morishima Gross Elasticity of Substitution

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- Assumed two inputs, capital and labor
- Value of ES indicates how relative input shares change when input prices change

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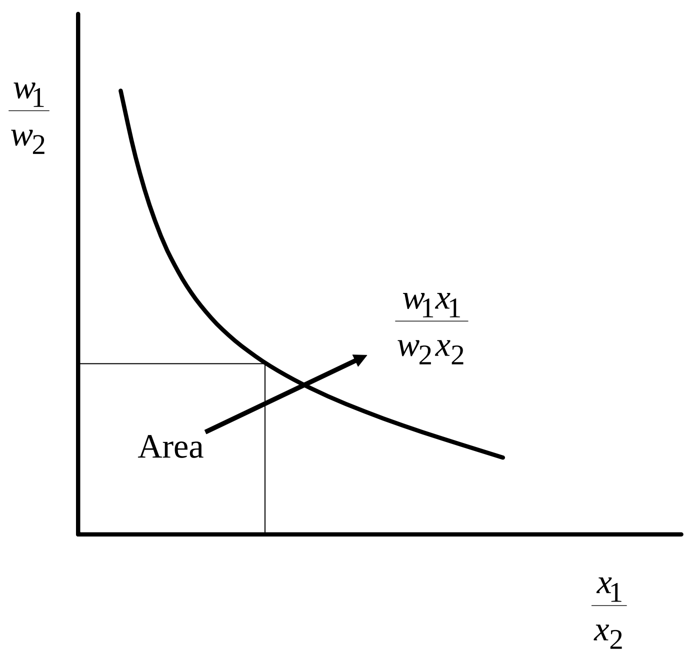
This measures the elasticity of the following curve.



Generalization to $n$ inputs

Allen-Uzawa Elasticity of Substitution (AUES)

$$\sigma_{ij}^{AU}(y, w) = \frac{c_{ij}(y, w)c(y, w)}{c_i(y, w)c_j(y, w)}$$

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Blackorby and Russell proposed an alternative:

The Morishima Elasticity of Substitution (MES) is given by

$$\sigma_{ij}^{M}(y,w) = \frac{w_i c_{ij}(y,w)}{c_j(y,w)} - \frac{w_i c_{ii}(y,w)}{c_i(y,w)}$$

Morishima (1967), Blackorby and Russell (1975).

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Morishima (1967), Blackorby and Russell (1975).

Note that

$$\begin{aligned} \sigma_{ij}^{M}(y,w) &= \frac{\partial x_j}{\partial w_i}\frac{w_i}{x_j} - \frac{\partial x_i}{\partial w_i}\frac{w_i}{x_i} \\ &= \varepsilon_{ji} - \varepsilon_{ii}. \end{aligned}$$

Bertoletti (2001, 2005) resurrected the

Hotelling-Lau Elasticity of Substitution (HLES)

$$\sigma_{ij}^{HL}(p, w) = -\frac{\pi_{ij}(p, w)\pi(p, w)}{\pi_i(p, w)\pi_j(p, w)}$$

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We consider the The Morishima Gross Elasticity of Substitution (MGES)

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and argue that it is a better gross ES than the HLES.

## An Illustrative Example:

$$y = f(x) = \left[\min\left\{x_1, g(x_2, x_3)\right\}\right]^b, \qquad 0 < b < 1,$$

where $g(\cdot)$ is homogeneous of degree one in $(x_2, x_3)$. The corresponding cost function is:

$$c(y, w) = \left[w_1 + e(w_2, w_3)\right] y^{1/b},$$

where

$$e(w_2, w_3) = \min\left\{w_2 x_2 + w_3 x_3 : g(x_2, x_3) \geq 1\right\}.$$

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If we let the aggregator function be Cobb-Douglas,

$$g(x_2, x_3) = x_2^a x_3^{1-a}$$

then

$$e(w_2, w_3) = \left(\frac{w_2}{a}\right)^a \left(\frac{w_3}{1-a}\right)^{1-a}$$

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$$\sigma_{23}^{AU}(y, w) = a^a (1-a)^{1-a} w_1 w_2^{-a} w_3^{a-1} + 1.$$

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Since the aggregator function is Cobb-Douglas, one would expect that the AUES for inputs 2 and 3 would be unity. However, the above AUES can take on any value between one and infinity as input prices vary for any $a \in (0, 1)$. This was the key feature in the example provided by Blackorby and Russell (1989). (They set $a = 1/2$.)

The profit function is

$$\pi(p, w) = \frac{B}{1+d}\left[p^{\frac{1}{1-b}}\right][w_1 + e(v)]^{1+d}$$

where

$$1 + d = -\frac{b}{1-b} \qquad \text{and} \qquad B = (1+d)\left[b^{\frac{b}{1-b}} - b^{\frac{1}{1-b}}\right].$$

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Hence, $\sigma_{23}^{HL}(p, w)$ and $\sigma_{23}^{AU}(y, w)$ differ even though, under homotheticity, they should be the same. Moreover,

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However, when the aggregator is Cobb-Douglas

$$\sigma_{23}^{HL}(w) = \frac{1-b}{b}\left[a^a(1-a)^{1-a}w_1 w_2^{-a} w_3^{a-1} + 1\right] - \frac{1}{b}$$

which can take any value from $-1/b$ to infinity for any $a \in (0,1)$ and $b \in (0,1)$.

Properties of the MGES

For the case of many inputs we calculate

$$
\begin{aligned}
-\frac{\partial \ln\left(\dfrac{x_i}{x_j}\right)}{\partial \ln\left(\dfrac{w_i}{w_j}\right)} &= \frac{\partial \ln x_j}{\partial \ln\left(\dfrac{w_i}{w_j}\right)} - \frac{\partial \ln x_i}{\partial \ln\left(\dfrac{w_i}{w_j}\right)} \\
&= \frac{\partial x_j}{\partial\left(\dfrac{w_i}{w_j}\right)}\frac{\left(\dfrac{w_i}{w_j}\right)}{x_j} - \frac{\partial x_i}{\partial\left(\dfrac{w_i}{w_j}\right)}\frac{\left(\dfrac{w_i}{w_j}\right)}{x_i} \\
&= \frac{\partial x_j}{\partial w_i}\frac{\partial w_i}{\partial\left(\dfrac{w_i}{w_j}\right)}\frac{\left(\dfrac{w_i}{w_j}\right)}{x_j} - \frac{\partial x_i}{\partial w_i}\frac{\partial w_i}{\partial\left(\dfrac{w_i}{w_j}\right)}\frac{\left(\dfrac{w_i}{w_j}\right)}{x_i} \\
&= \frac{\partial x_j}{\partial w_i} w_j \frac{\left(\dfrac{w_i}{w_j}\right)}{x_j} - \frac{\partial x_i}{\partial w_i} w_j \frac{\left(\dfrac{w_i}{w_j}\right)}{x_i} \\
&= \frac{\partial x_j}{\partial w_i}\frac{w_i}{x_j} - \frac{\partial x_i}{\partial w_i}\frac{w_i}{x_i}
\end{aligned}
$$

Using Hotelling's Lemma:

$$x_j = \frac{-\partial \pi(p, w)}{\partial w_i} \text{ and } \frac{\partial x_j}{\partial w_i} = \frac{-\partial^2 \pi(p, w)}{\partial w_i \partial w_j}$$

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we get

$$\begin{aligned}
\frac{\partial x_j}{\partial w_i}\frac{w_i}{x_j} - \frac{\partial x_i}{\partial w_i}\frac{w_i}{x_i} &= w_i\left(\frac{-\pi_{ij}}{-\pi_j} - \frac{-\pi_{ii}}{-\pi_i}\right) \\
&= w_i\left(\frac{\pi_{ij}}{\pi_j} - \frac{\pi_{ii}}{\pi_i}\right) \\
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The MGES provides immediate information about how the income ratio, $S_{ij}(p,w) = w_i x_i^* / w_j x_j^*$, changes with a change in the input price ratio:

$$\frac{\partial \ln S_{ij}(p,w)}{\partial \ln\left(\dfrac{w_i}{w_j}\right)} = 1 - \sigma_{ij}^{MG}.$$

It is also interesting to derive the relationship between the MES and the MGES. Let $x = h(y, w)$ and $x = x(p, w)$ be the cost-minimizing and profit-maximizing choices for the input vector. Also let $y = y(p, w)$ be the profit-maximizing output. Then

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Differentiate with respect to $w_j$ to get

$$\frac{\partial x_i(p, w)}{\partial w_j} = \frac{\partial h_i(y, w)}{\partial w_j} + \frac{\partial h_i(y, w)}{\partial y}\frac{\partial y(p, w)}{\partial w_j}.$$

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Invoking Hotelling/Shephard, we obtain

$$\pi_{ij}(p, w) = -\left[c_{ij}(y, w) + c_{iy}(y, w)\pi_{pj}(p, w)\right]$$

and

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The Morishima gross elasticity of substitution is defined by

$$\sigma_{ij}^{MG}(p, w) = w_i\left(\frac{\pi_{ij}(p, w)}{\pi_j(p, w)} - \frac{\pi_{ii}(p, w)}{\pi_i(p, w)}\right).$$

Thus, the MGES can be written as a function of the MES as follows:

$$
\begin{aligned}
&\sigma_{ij}^{MG}(p,w) \\
&= \frac{-w_i\left[c_{ij}(y,w)+c_{iy}(y,w)\pi_{pj}(p,w)\right]}{\pi_j(p,w)} \\
&- \frac{-w_i\left[c_{ii}(y,w)+c_{iy}(y,w)\pi_{pi}(p,w)\right]}{\pi_i(p,w)} \\
&= w_i\left(\frac{c_{ij}(y,w)}{c_j(y,w)}-\frac{c_{ii}(y,w)}{c_i(y,w)}\right) \\
&+ w_i c_{iy}(y,w)\left(\frac{\pi_{pj}(p,w)}{\pi_j(p,w)}-\frac{\pi_{pi}(p,w)}{\pi_i(p,w)}\right) \\
&= \sigma_{ij}^{M}(y,w)+w_i c_{iy}(y,w)\left(\frac{\pi_{pj}(p,w)}{\pi_j(p,w)}-\frac{\pi_{pi}(p,w)}{\pi_i(p,w)}\right). \qquad (1)
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&= w_i \left( \frac{c_{ij}(y, w)}{c_j(y, w)} - \frac{c_{ii}(y, w)}{c_i(y, w)} \right) \\
&+ w_i c_{iy}(y, w) \left( \frac{\pi_{pj}(p, w)}{\pi_j(p, w)} - \frac{\pi_{pi}(p, w)}{\pi_i(p, w)} \right) \\
\\
&= \sigma_{ij}^{M}(y, w) + w_i c_{iy}(y, w) \left( \frac{\pi_{pj}(p, w)}{\pi_j(p, w)} - \frac{\pi_{pi}(p, w)}{\pi_i(p, w)} \right). \qquad (1)
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We now show that the MGES and the MES are equal if and only if the production function is homothetic. It is apparent from (1) that the Morishima gross elasticity of substitution is equal to the Morishima (net) elasticity of substitution for all input pairs if and only if

$$\frac{\pi_{pj}(p, w)}{\pi_j(p, w)} - \frac{\pi_{pi}(p, w)}{\pi_i(p, w)} = 0, \quad i, j = 1, ..., n.$$

This is equivalent to the condition that

$$\begin{aligned}\frac{\partial}{\partial p}\left(\frac{\pi_i(p,w)}{\pi_j(p,w)}\right) &= \frac{\pi_j(p,w)\pi_{ip}(p,w) - \pi_i(p,w)\pi_{jp}(p,w)}{[\pi_j(p,w)]^2} \\ &= \frac{\pi_j(p,w)\pi_{pi}(p,w) - \pi_i(p,w)\pi_{pj}(p,w)}{[\pi_j(p,w)]^2} \\ &= 0, \quad i,j = 1,...,n.\end{aligned}$$

This is the well-known condition for separability of input prices from the output price. This separability condition is equivalent to homotheticity of the production function. Not surprisingly, for our example, the MES and MGES are equal.