1. Introduction

Separability, as discussed here, refers to certain restrictions on functional representations of consumer (or social) preferences or producer technologies. These restrictions add structure to the decision making tasks undertaken by economic agents. They also allow the economic researcher to study the behavior of these agents in a more effective manner.

To keep things simple we will focus on consumers. Consumers must make choices among a large number of goods—both consumption and leisure, both present and future. The apparent complexity of their decision-making problems may lead consumers to engage in simplified budgeting practices. Are these procedures consistent with rational behavior? In modeling consumer behavior, economists— theoretical and empirical—employ models that consider only a subset or several subsets of the complete list of goods and services consumed. Can these practices be justified? And, if so, what kinds of restrictions do these rationalizations place on the preferences and the behavior of the consumer. In what follows, we discuss several proposals that address this class of problems and present their solutions.

One way to think about reducing the complexity of the allocation decision is to imagine that the consumer receives a lump sum of money that he or she first allocates to broad classes of commodities, such as food, shelter, and recreation. Detailed decisions about how to spend the money that has been allocated to the food budget are postponed until one is actually in the store buying specific food items.

More formally, if the correct amount of money to spend on food commodities, for example, has been allocated, under what circumstances is the consumer able to dispense the food budget among food commodities knowing only the food prices? If the consumer can arrive at an optimal pattern
of food expenditures in this way, preferences are said to be *decentralizable*. It is fairly obvious that, if food commodities were separable from all other commodities, decentralization would be possible. What is somewhat surprising, however, is that separability is necessary as well as sufficient for this practice to be rationalized. We first present more formally the concepts of decentralization and separability and then discuss their equivalence.

Consumption bundles are \( N \)-tuples, \( x = (x_1, \ldots, x_N) \), that are elements of a consumption space, \( \Omega \). We take \( \Omega \) to be a closed and convex subset of \( \mathbb{R}^N_+ \). Thus we begin with a consumer with a well-defined, neoclassical utility function, \( U : \Omega \to \mathbb{R} \). Partition the set of variable indices, \( I = \{1, 2, \ldots, N\} \), into two subsets \( \{I^1, I^2\} \). This divides the set of goods into two groups, one and two, and we can write the consumption space as \( \Omega = \Omega^1 \times \Omega^2 \) and the consumption vector as \( x = (x^1, x^2) \). The consumer faces commodity prices given by \( p = (p_1, \ldots, p_N) \) and allocates income \( y \) among the \( N \) goods. The price vector can also be written as \( p = (p^1, p^2) \).

The consumer’s utility maximization problem,

\[
\max_x U(x) \quad \text{subject to} \quad p \cdot x = y,
\]

can be rewritten as

\[
\max_{x^1, x^2} U(x^1, x^2) \quad \text{subject to} \quad p^1 \cdot x^1 + p^2 \cdot x^2 = y. \tag{2}
\]

The solution to (1) [or (2)] is denoted by

\[
x^* = (\Phi^1(y, p), \Phi^2(y, p)). \tag{3}
\]

Expenditure on goods in group two is denoted by \( y_2 \). Group two expenditure is optimal if \( y_2 = p^2 \cdot \Phi^2(y, p) \).

2. Decentralization and Separability

The consumer’s utility maximization problem may be decentralized if the consumer is able to optimally allocate expenditure on group-two commodities knowing only group-two optimal expenditure and group-two prices. More formally, the utility maximization problem is *decentralizable* (for group two) if there exists a function \( \phi^2 \) such that

\[
\phi^2(y_2, p^2) = \Phi^2(y, p) \quad \text{if} \quad y_2 = p^2 \cdot \Phi^2(y, p).
\]
Now the question is: what restriction on the consumer’s utility function allows for decentralizability of the utility maximization problem? As it turns out, decentralizability for group two goods is possible if and only if group two goods are separable from group one goods. We turn to the formal definition of separability.

According to the original Leontief-Sono definition of separability, the goods in group two are separable from the goods in group one if
\[ \frac{\partial}{\partial x_k} \left( \frac{\partial U(x)/\partial x_i}{\partial U(x)/\partial x_j} \right) \equiv 0, \]  
for all \( i, j \in I^2, k \in I^1 \). See Leontief (1947a, 1947b) and Sono (1945, 1961).

The condition (4) says that marginal rates of substitution between pairs of goods in group two are independent of quantities in group one and hence that an aggregator function exists for group-two goods. Under fairly mild conditions, this aggregator function may be defined by
\[ U^2(x^2) := U \left( O^1, x^2 \right) \]
for any choice of a fixed vector, \( O^1 \in \Omega^1 \); i.e., \((O^1, x^2) \in \Omega\). Different choices of \( O^1 \) give rise to different aggregator functions that are ordinally equivalent. Having accomplished this, we also define a macro function, \( \mathbb{U} \), that is defined by
\[ U(x^1, x^2) = \mathbb{U} \left( x^1, U^2(x^2) \right) . \]

EXAMPLE: Let \( I = \{1, 2, 3, 4\}, I^1 = \{1, 2\}, \) and \( I^2 = \{3, 4\} \). Consider the utility function given by
\[ U(x_1, x_2, x_3, x_4) = x_1^{1/3} x_3^{1/3} x_4^{1/3} + x_2^{1/2} x_3^{1/4} x_4^{1/4} . \]
By defining \( u_2 = U^2(x_3, x_4) = x_3^{1/3} x_4^{1/3} \) and \( \mathbb{U}(x_1, x_2, u_2) = x_1^{1/3} u_2 + x_2^{1/2} u_2^{3/4} \), we observe that
\[ U(x_1, x_2, x_3, x_4) = \mathbb{U}(x_1, x_2, u_2) , \]
and hence that group two is separable from group one. One may also confirm this fact using the Leontief-Sono conditions. The reader should notice that the choice of the aggregator and macro function is not unique. An alternative choice would be \( u_2 = U^2(x_3, x_4) = x_3^{1/2} x_4^{1/2} \) and \( \mathbb{U}(x_1, x_2, u_2) = x_1^{1/3} u_2^{2/3} + x_2^{1/2} u_2^{1/2} \). In this case, the aggregator function is chosen to be homogeneous of degree one. The reader should also notice that group one is not separable from group two; separability is not a symmetric concept.
The formal result on decentralizability can now be stated. The consumer’s utility maximization problem is decentralizable for group two if and only if group two is separable from group one—i.e., if and only the utility function can be written as

\[ U(x^1, x^2) = \bigcup \bigcup x^1, U^2 (x^2) \bigcup. \]

The implication of this result is that the consumer utility maximization problem may be broken up into two parts:

1. Solve

\[ \max_{x^2} U^2 (x^2) \quad \text{subject to} \quad p^2 x^2 \leq y_2 \]  \hspace{1cm} (6)


to get the conditional demand function, \( \phi^2 (y_2, p^2) \), for group two.

2. Solve

\[ \max \bigcup x^1, U^2 \bigcup \phi^2 (y_2, p^2) \bigcup \bigcup \quad \text{subject to} \quad p^1 x^1 + y_2 \leq y \]  \hspace{1cm} (7)


to get the optimal demands for group one and the optimal income allocation for group two.

While separability is inherently an asymmetric concept we may want to consider the symmetric case. We may also want to extend the analysis to consider \( R \) groups of goods. To this end, let \( \{ I^1, ..., I^R \} \) be a partition of the original set of variable indices \( \mathcal{I} \). The goods vector and the price vector can be written as \( x = (x^1, ..., x^R) \) and \( p = (p^1, ..., p^R) \), respectively. Then the utility function is separable in the partition \( \{ I^1, ..., I^R \} \) if and only if the utility function may be written as

\[ U(x^1, ..., x^R) = \bigcup U^1 (x^1), ..., U^R (x^R) \bigcup. \]  \hspace{1cm} (8)

Expenditure on goods in group \( r \) is denoted by \( y_r, r = 1, ..., R \). Group \( r \) expenditure is optimal if \( y_r = p^r \cdot \Phi^r (y, p) \).

The consumer’s utility maximization problem may be decentralized if the consumer is able to optimally allocate expenditure on group \( r \) commodities knowing only group \( r \) optimal expenditure and group \( r \) prices. More formally, the utility maximization problem is \textit{decentralizable} for the partition \( \{ I^1, ..., I^R \} \) if there exist functions \( \phi^r, r = 1, ..., R \), such that

\[ \phi^r (y_r, p^r) = \Phi^r (y, p) \quad \text{if} \quad y_r = p^r \cdot \Phi^r (y, p), \quad r = 1, ..., R. \]
A useful device for explicating the necessary and sufficient conditions for decentralizability of the utility maximization problem is the conditional indirect utility function. For the $R$ group case it is defined as

$$H(y_1, ..., y_R, p) = \max_x \{U(x) : p^r x^r \leq y^r, r = 1, ..., R\}.$$  \hspace{1cm} (9)

This function yields the maximum utility conditional on an income allocation, $(y_1, ..., y_R)$, among the $R$ groups. In a second stage one can solve for the optimal income allocation by solving

$$\max_{y_1, ..., y_R} H(y_1, ..., y_R, p) \text{ subject to } \sum_{r=1}^{R} y^r \leq y. \hspace{1cm} (10)$$

Denote the solution to (10) by the expenditure-allocation functions

$$y_r = \theta^r (y, p), r = 1, ..., R.$$  \hspace{1cm} (11)

A remarkable feature of the conditional indirect utility function is that

$$H (\theta^1 (y, p), ..., \theta^R (y, p), p) = W(y, p) = \max_x \{U(x) : p \cdot x \leq y\}$$

Thus, the overall consumer utility maximization may always be performed in the two steps given in (9) and (10) even in the absence of separability restrictions on the utility function.

The results for decentralizability in the $R$-group symmetric case can be stated in two parts. First, the utility function is separable in the partition \{I_1, ..., I_R\} as in (8) if and only if the conditional indirect utility function may be written as

$$H(y_1, ..., y_R, p) = U (v^1 (y_1, p^1), ..., v^R (y_R, p^R))$$

where

$$v^r (y_r, p^r) = \max_{x^r} \{U^r (x^r) : p^r x^r \leq y^r\}, \hspace{1cm} r = 1, ..., R,$$  \hspace{1cm} (12)

are the conditional indirect utility functions for the aggregator functions, $U^r (x^r), r = 1, ..., R$. Second, the consumer’s utility maximization problem is decentralizable for the partition \{I_1, ..., I_R\} if and only if the utility function is separable in the partition \{I_1, ..., I_R\}.
Separability in the partition \( \{ I^1, ..., I^R \} \) allows the following two-step budgeting procedure. In the first step the consumer solves the problem in (12), which yields conditional demand functions \( x^r = \phi^r (y^r, p^r), r = 1, ..., R \). In the second step, the consumer solves (10), which yields the optimal income allocation. While the first step economizes the informational requirements—knowledge of only in-group prices are needed—the second step requires the entire vector of prices. However, additional restrictions on the utility function will reduce both the computational and informational burden of the second step; these restrictions will be discussed shortly.

Decentralization may be possible in more than one partition of the set of commodities. Moreover, the separable sectors might well overlap. Suppose, for example, that \( U \) is a social welfare function and \( x^r \) is the consumption vector of consumer \( r \). As some policies affect only a subset of the population, one might want to analyze separately the social welfare of, say, two subgroups of the population: (1) a particular ethnic group and (2) those in a particular geographical region. Clearly, the two groups could overlap. A deep result of Gorman [1968] indicates that the existence of such overlapping separable groups has powerful implications for the structure of the welfare function.

3. Additive Structures

Let \( I^r \) and \( I^c \) be the two (separable) groups of interest and suppose that \( I^1 := I^r - I^c \neq \emptyset, I^2 := I^r \cap I^c \neq \emptyset, \) and \( I^3 := I^c - I^r \neq \emptyset \). Let \( I^4 \) be the complement of \( I^r \cup I^c \) in In \( I \). Gorman’s overlapping theorem indicates that \( I^1, I^2, \) and \( I^3 \) are also separable from their complements in \( I \). One of the important aspects of Gorman’s overlapping theorem is that this structure is equivalent to the following representation of \( U \):\(^1\)

\[
U(x) = U \left( U^1(x^1) + U^2(x^2) + U^3(x^3), x^4 \right).
\]  

Note that what drives this result on (groupwise) additive structures is not simply separability of each of the sectors, \( I^1, I^2, \) and \( I^3, \) from their complements but also separability of arbitrary unions of these sectors. This observation suggests the general result on groupwise-additive structures (Debreu [1960] and Gorman [1968]):

\(^1\)Additional regularity conditions on the “essentiality” of each group are required for this result (see Gorman [1968] or Blackorby, Primont, and Russell [1978, 1998]).
\[ U(x) = \left( \sum_{r=1}^{R} U^r(x^r) \right), \quad R > 2, \quad (14) \]

if and only if arbitrary unions of subsets of the partition \( \{I^1, \ldots, I^R\} \) are separable from their complements. Note that a special case of (14) is when each element of the partition \( I \) is a singleton \((R = n)\), in which case,

\[ U(x) = \left( \sum_{i=1}^{N} U^i(x_i) \right), \quad N > 2. \quad (15) \]

That is, \( u \) is additive in the variable themselves.

Another example in the context of social welfare functions illustrates the power of Gorman’s overlapping theorem. Suppose that the social decision rule satisfies the anonymity condition: only the individuals’ utilities and not their names should matter in social evaluation. This standard social choice assumption is equivalent to symmetry of the social welfare function: permuting the names of individuals does not affect the value of the function. Let \( \{1, 2\} \) be the subset of individuals who are affected by some set of policies, and suppose that these policies are judged solely by their effects on these two individuals. This implies that \( \{1, 2\} \) is separable from its complement in the set of citizens and hence that

\[ W(u_1, \ldots, u_R) = J(F(u_1, u_2), u_3, \ldots, u_R) \quad (16) \]

The fact that \( W \) is symmetric means that it can be written as

\[ J(F(u_1, u_2), u_3, \ldots, u_R) = J(F(u_1, u_3), u_2, u_4, \ldots, u_R) \quad (17) \]

This in turn implies that \( \{1,3\} \) is separable from its complement in the set of citizens. However, these sets have a nonempty intersection, \( \{2\} \), and the overlapping theorem implies that the social welfare function can be written as

\[ W(u_1, \ldots, u_R) = \mathbb{I} \left( f^1(u_1) + f^2(u_2) + f^3(u_3), u_4, \ldots, u_R \right) \]

Proceeding by induction yields a strong implication: the social welfare function must be additive:

\[ W(u_1, \ldots, u_R) = W \left( f^1(u_1) + \cdots + f^R(u_R) \right) \]
The Gorman overlapping argument establishing the groupwise additive structure (13) requires the existence of at least two overlapping separable sets resulting in at least three separable sets. Put differently, separability of arbitrary unions of nonoverlapping sets in binary partition \( \{I^1, I^2\} \) adds no restriction to separability of the two sets from their complements. But two-group additivity arises in many contexts. One example is the typical additive utility function in overlapping-generations models where each generation’s finite lifetime is divided into two periods (often a “work” period and a “retirement” period). A special case of two-group additivity is the quasi-linear utility function that is so critical to the analysis of public goods and incentive compatibility. The stronger conditions required for two-group additivity are based on Sono’s [1945, 1961] independence condition. A set of variables, \( I^1 \), is said to be \textit{Sono independent} of \( I^2 \) if there exist functions \( \psi_{ji}, i, j \in I^1 \), such that

\[
\frac{\partial}{\partial x_i} \left( \ln \frac{\partial U(x)}{\partial x_k} \right) = \frac{\partial}{\partial x_i} \left( \ln \frac{\partial U(x)}{\partial x_j} \right) = \psi_{ji}(x^1) \\
\forall i, j \in I^1, \forall k, \ell \in I^2
\]

That is, the effect of changing the consumption of a commodity in sector 1 on the marginal rates of substitution between pairs of variables, one in sector \( I^1 \) and the other in sector \( I^2 \), depends only on the values of consumption levels in sector 1.

The necessary and sufficient condition for two-group additivity is as follows: \( U \) can be written as

\[
U(x) = U \left( U^1 (x^1) + U^2 (x^2) \right)
\]

if and only if \( I^1 \) is independent of \( I^2 \) and is separable from \( I^2 \). Although this structure is commonly referred to as “separable,” it is clearly stronger than separability as the term has historically been used.

4. Two Stage Budgeting

It was emphasized in Section 2 that the principal motivation for separability is that it rationalizes decentralizability of the (possibly complex)
expenditure-constrained maximization problem. But, since separability only guarantees that sectoral expenditure is optimally allocated within the sector, full rationalization requires in addition that the optimal amount of money be allocated to each sector. This is accomplished by solving the allocation problem (10).

But solving this problem is tantamount to solving the overall optimization problem and does not seem to reduce the informational requirements. For this reason, Strotz [1957] and Gorman [1959] emphasized the existence of sectoral price aggregates (indexes) that can be used to simplify the first stage of the two-stage optimization problem. Formally, we say that price aggregation is possible if the allocation functions in (11) can be written as

$$y_r = \Theta^r (y, \Pi^1 (p^1), ..., \Pi^R (p^R)), r = 1, ..., R$$

(18)

(where the price-index functions, $\Pi^r, r = 1, ..., R,$ are not assumed to be homogeneous of degree one or even homothetic.)

Price aggregation is equivalent to the following structure for the direct utility function (in an appropriate permutation of the sectoral indices, 1, ..., $R$):

$$U(x) = F \left( \sum_{r=1}^{D} f^r(x^r) + G \left( f^{D+1}(x^{D+1}), ..., f^{R}(x^R) \right) \right)$$

(19)

where each $f^r, r = D + 1, ..., R,$ is homothetic and hence can be normalized to be homogeneous of degree one and the sectoral indirect utility functions dual to the first $D$ aggregator functions, defined by

$$v^r (y_r, p^r) = \max_{x^r} \left\{ U^r (x^r) : p^r x^r \leq y_r \right\}, \quad r = 1, ..., R,$$

(20)

have the structure,

$$v^r (y_r, p^r) = v^r \left( \frac{y_r}{\Pi^r (p^r)} \right) + w^r (p^r), \quad r = 1, ..., D,$$

(21)

where each $w^r$ is homogeneous of degree zero in $p^r$ and $\Pi^r$ is homogeneous of degree one in $p^r.$

\textsuperscript{3}Gorman [1959], assuming away the troublesome two-group case, proved a restricted version of this result. Exploiting Sono independence and some newer results not available to Gorman in 1959, Blackorby and Russell [1997] proved the general result, showing that the entire structure needed for two-stage budgeting is, in fact, imbedded in the two-group case.
An interesting special case of (19) is obtained if $D = 0$:

$$U(x) = F(f^1(x^1), ..., f^R(x^R))$$  \hfill (22)

In this case, the first stage of the budgeting algorithm can be expressed as choosing $X^1, ..., X^R$ to maximize $F(X^1, ..., X^R)$ subject to the budget constraint, $\sum_r II^r(p^r)X^r = y$. This structure allows the two-stage budgeting to be accomplished using only price and quantity aggregates in the first stage.

5. Closing Remarks

We have chosen consumer demand theory as a way to illustrate the use of separability in economic analysis. However, separability assumptions, either explicit or implicit, are found in numerous areas in economics. For references to the literature, the reader can consult Blackorby, Primont, and Russell [1978, 1998].

References


